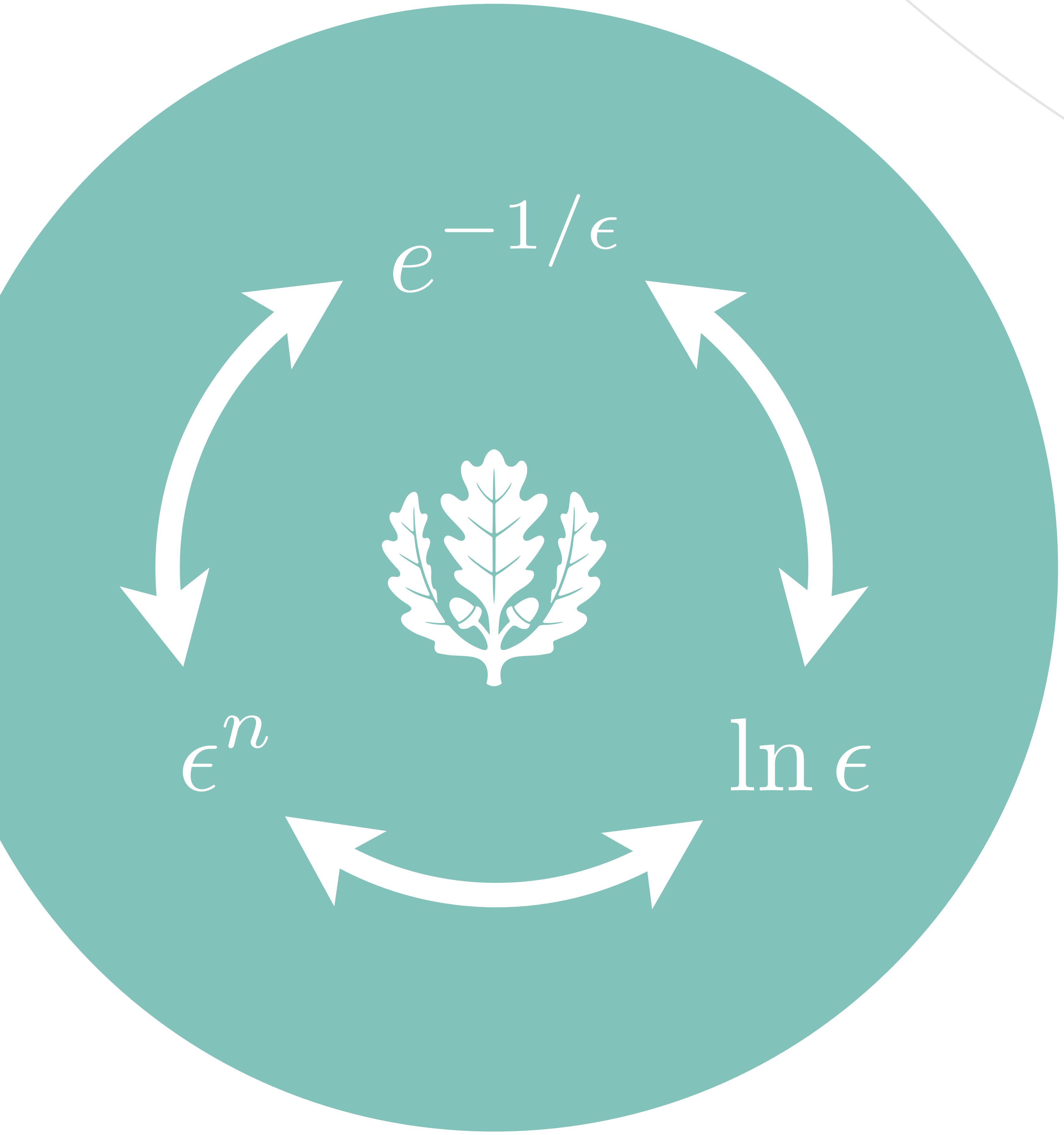

A Brief Introduction to Perturbation Theory

Mini-lecture series

Zachary Harris

November 2nd, 2022

Overview

Lecture 1:

How do we use perturbation theory to solve hard problems?

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots$$

Lecture 2:

How can we extract meaningful information from divergent series?

$$\sum_{n=0}^{\infty} a_n \epsilon^n \stackrel{?}{=} \infty$$

Lecture 3:

How can we learn about non-perturbative physics from perturbation theory?

$$e^{-1/x} \stackrel{?}{=} 0 + 0 \cdot x + 0 \cdot x^2 + \ldots$$

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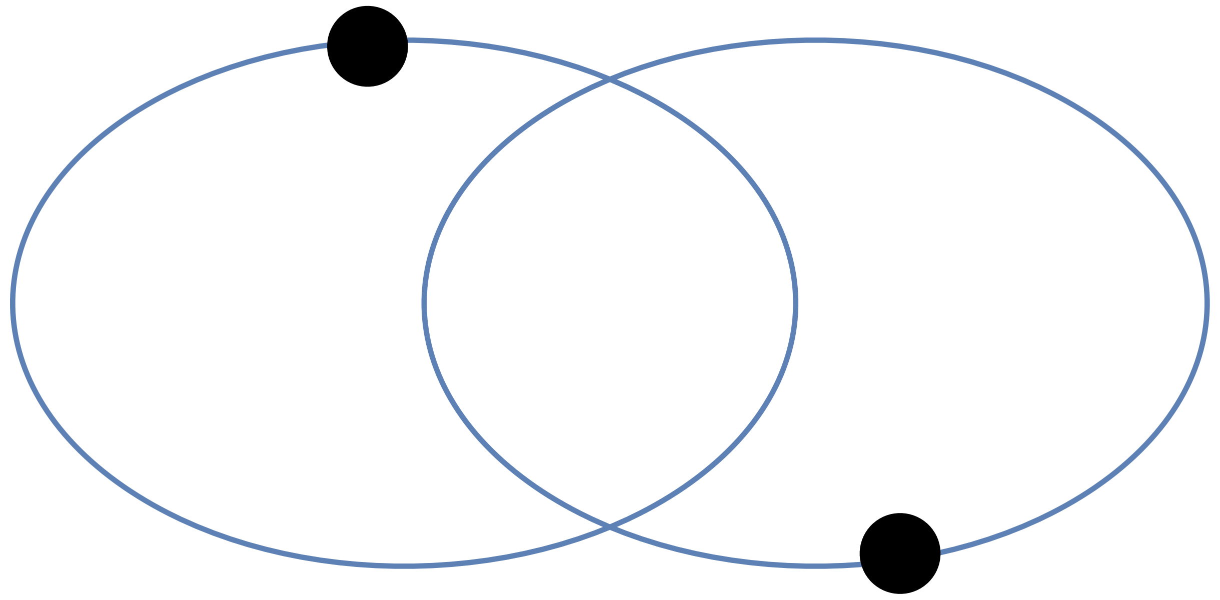
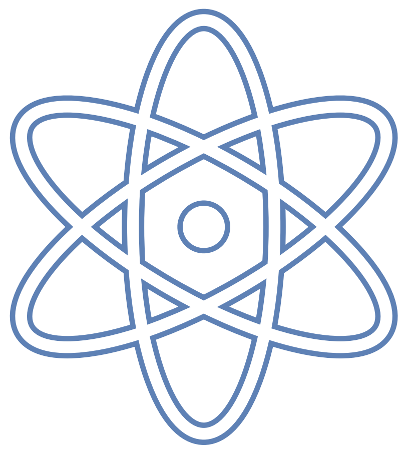
Hard problems

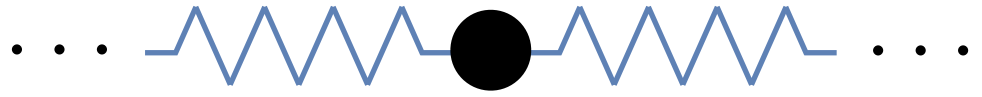
Physics classes → exactly solvable problems

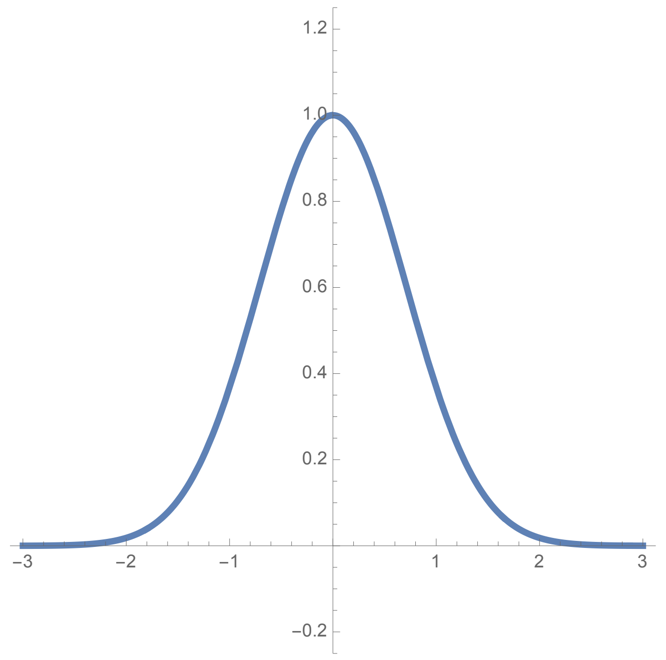
$$V(r) = \frac{1}{r}$$





$$V(x) = x^2$$





Physics research → most interesting problems have no exact solution

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi\,, \qquad R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu}\,, \qquad Z[J] = \mathcal{N}\int \mathcal{D}\Phi \exp\left(\frac{i}{\hbar}S[\Phi, J]\right), \qquad \ldots$$

Approaches

Numerics:

Perturbation Theory:

1. Insert a small parameter

$$\text{HP} \;\rightarrow\; \text{HP}(\epsilon)$$

2. Write the answer as a power series

$$\text{ANS}(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$$

3. Set $\epsilon = 1$ and sum the series

$$\text{ANS} = \sum_{n=0}^{\infty} a_n$$

A familiar example

$$-y'' + \left(\frac{x^2}{4} + \frac{x^4}{4} - E\right)y = 0, \qquad y(\pm\infty) = 0$$

"Anharmonic oscillator"

This is a hard problem → perturbation theory!

STEP 1: $$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} + \frac{x^4}{4}\right)y(x) = Ey(x) \qquad \Longrightarrow \qquad \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} + \epsilon\frac{x^4}{4}\right)y(x;\epsilon) = E(\epsilon)y(x;\epsilon)$$

Unperturbed problem ($\epsilon = 0$) must be exactly solvable:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4}\right)y(x) = Ey(x) \qquad \Longrightarrow \qquad E = \frac{1}{2}, \quad y(x) = \frac{e^{-x^2/4}}{\sqrt[4]{2\pi}}$$

"Harmonic oscillator"

Anharmonic oscillator

STEP 2:

$$E(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n = \frac{1}{2} + a_1\epsilon + a_2\epsilon^2 + \dots$$

$$y(x;\epsilon) = \sum_{n=0}^{\infty} y_n(x)\epsilon^n = \frac{e^{-x^2/4}}{\sqrt[4]{2\pi}} + y_1(x)\epsilon + y_2(x)\epsilon^2 + \dots$$

Plug in and match powers of ϵ:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} + \epsilon\frac{x^4}{4}\right)\left(\frac{e^{-x^2/4}}{\sqrt[4]{2\pi}} + y_1(x)\epsilon + \dots\right) = \left(\frac{1}{2} + a_1\epsilon + \dots\right)\left(\frac{e^{-x^2/4}}{\sqrt[4]{2\pi}} + y_1(x)\epsilon + \dots\right)$$

$$\epsilon^0: \quad \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4}\right)e^{-x^2/4} = \frac{1}{2}e^{-x^2/4}$$ ✓

$$\epsilon^1: \quad \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{x^2}{4} - \frac{1}{2}\right)y_1(x) = \left(a_1 - \frac{x^4}{4}\right)e^{-x^2/4} \quad \Rightarrow \quad a_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2/4}\left(\frac{x^4}{4}\right)e^{-x^2/4}$$

$$a_1 = \frac{3}{4}$$

$\vdots$

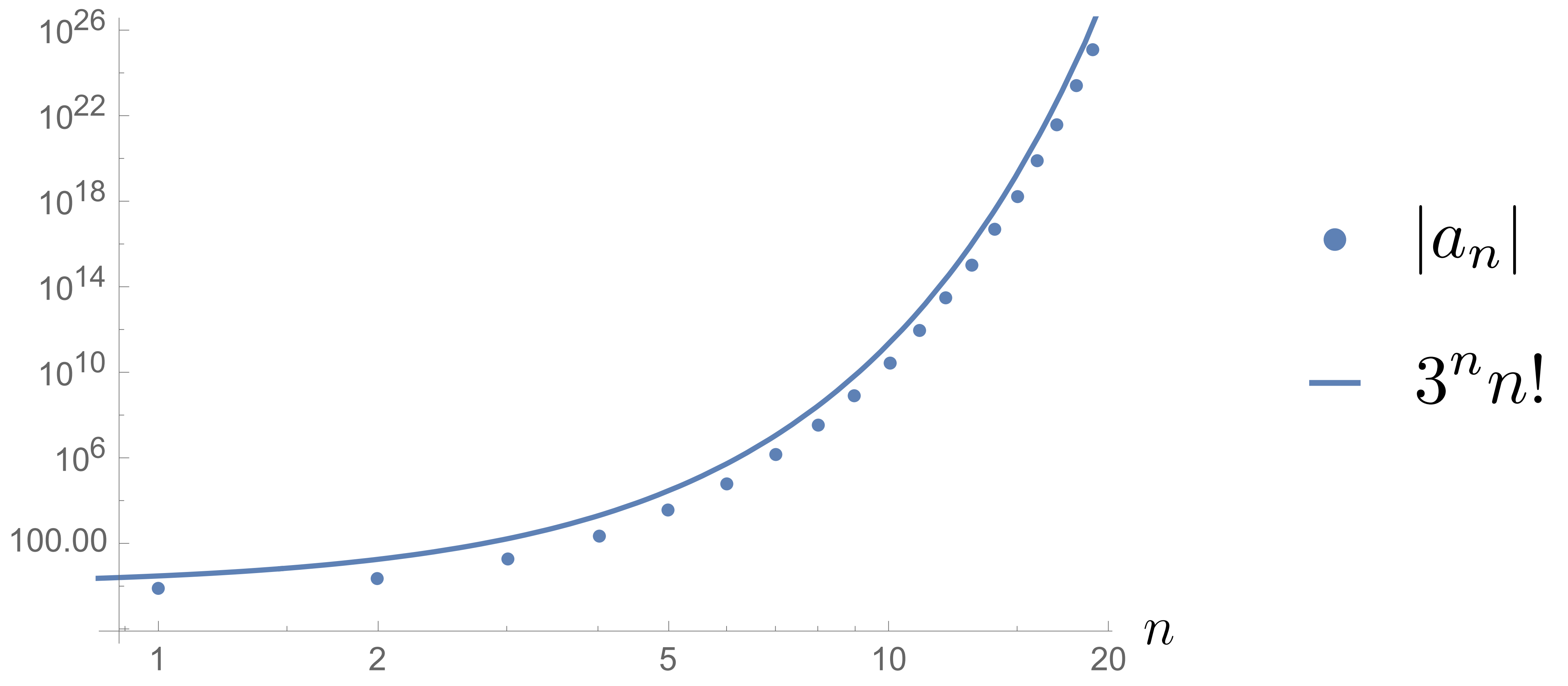
Beyond first order

This procedure is algorithmic for polynomial potentials (see BenderWu MATHEMATICA package)

$$E(\epsilon) = \frac{1}{2} + \frac{3}{4}\epsilon - \frac{21}{8}\epsilon^2 + \frac{333}{16}\epsilon^3 - \frac{30885}{128}\epsilon^4 + \frac{916731}{256}\epsilon^5 - \frac{65518401}{1024}\epsilon^6 + \ldots$$

STEP 3: $E = E(1)$

$$= \frac{1}{2} + \frac{3}{4} - \frac{21}{8} + \frac{333}{16} - \frac{30885}{128} + \frac{916731}{256} - \frac{65518401}{1024} + \ldots = ?$$



$E(\epsilon)$ is **divergent**

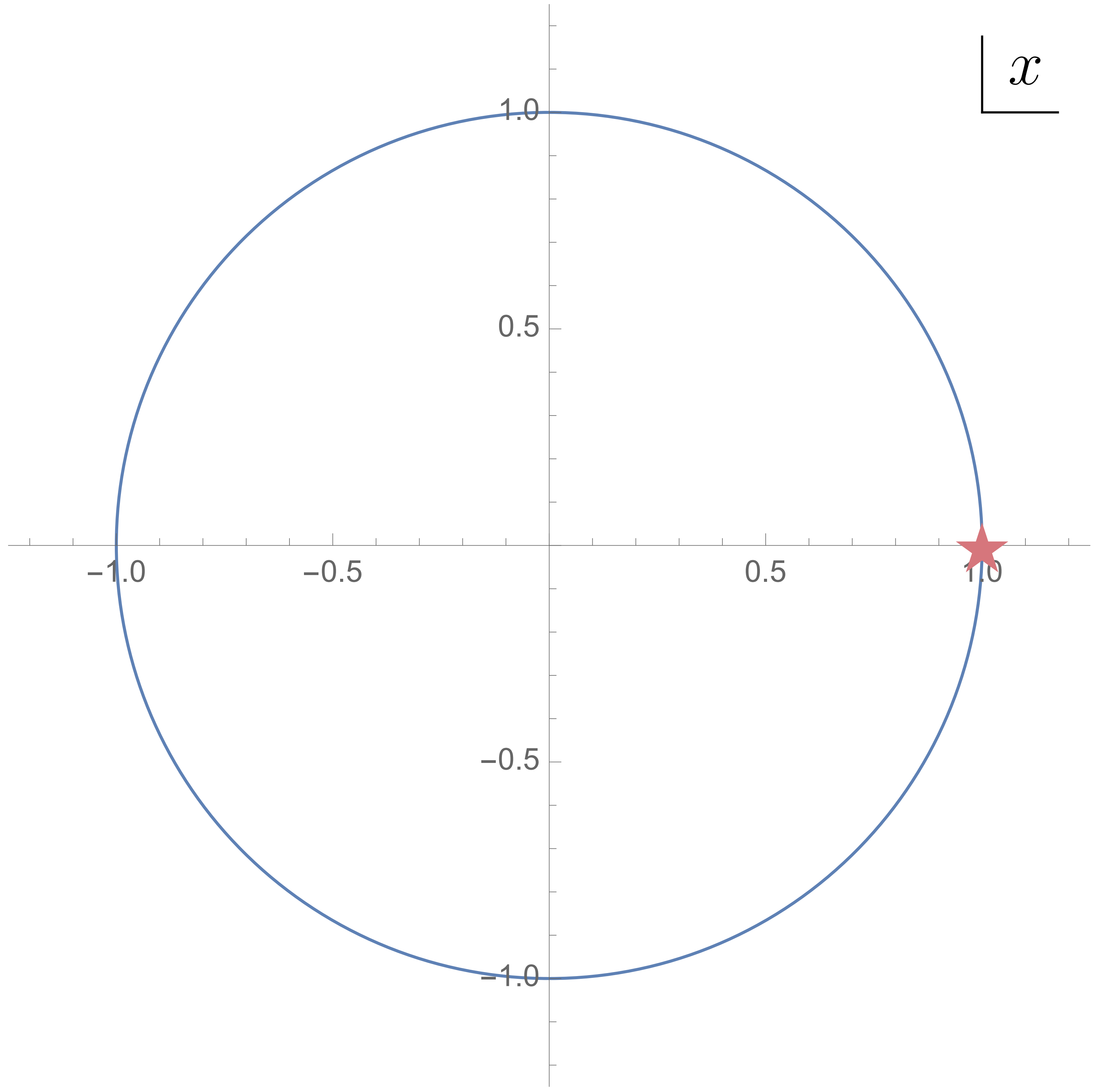
for all $\epsilon \neq 0$

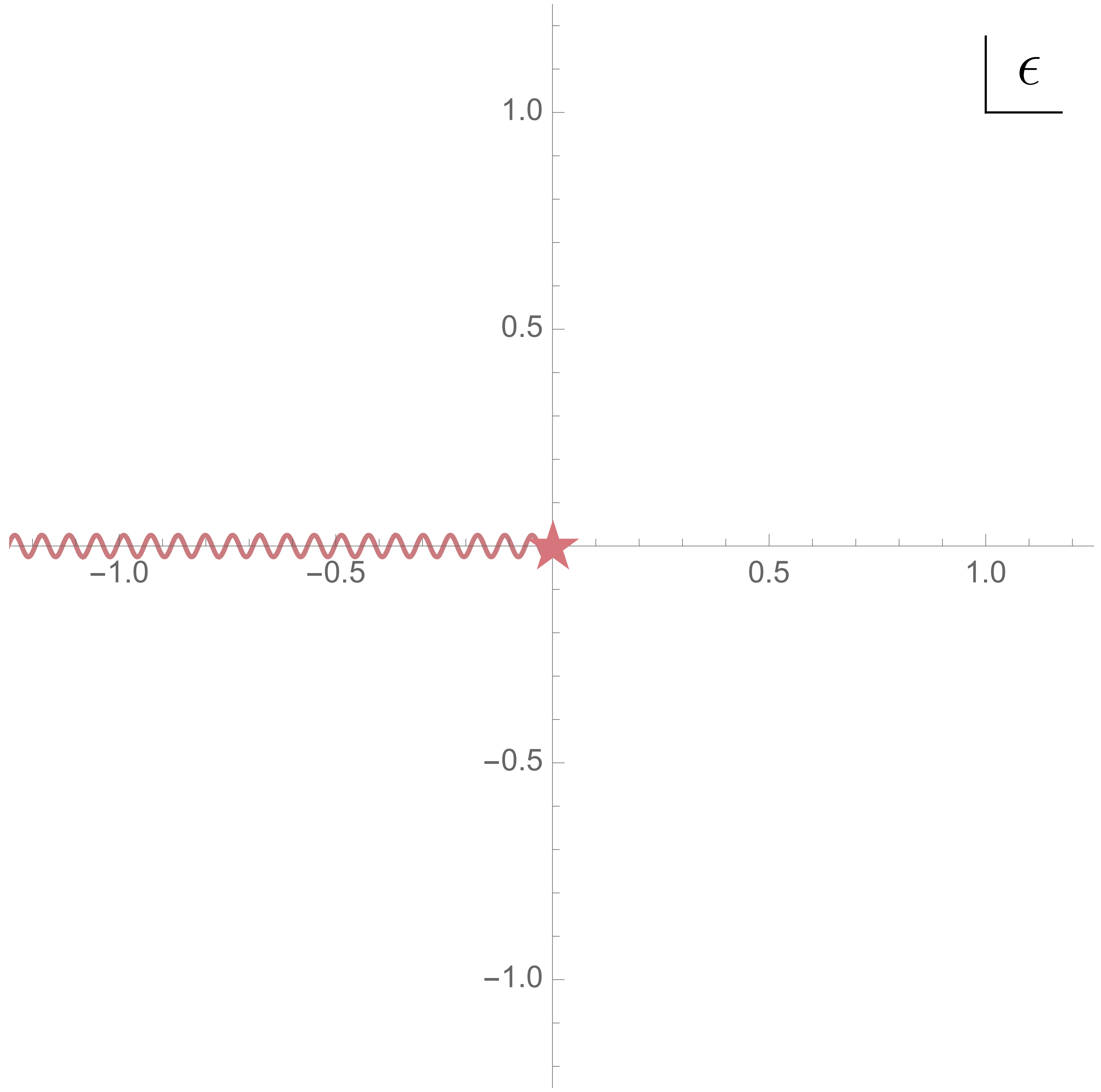
Singularities

Convergence is controlled by the singularity structure in the complex plane

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

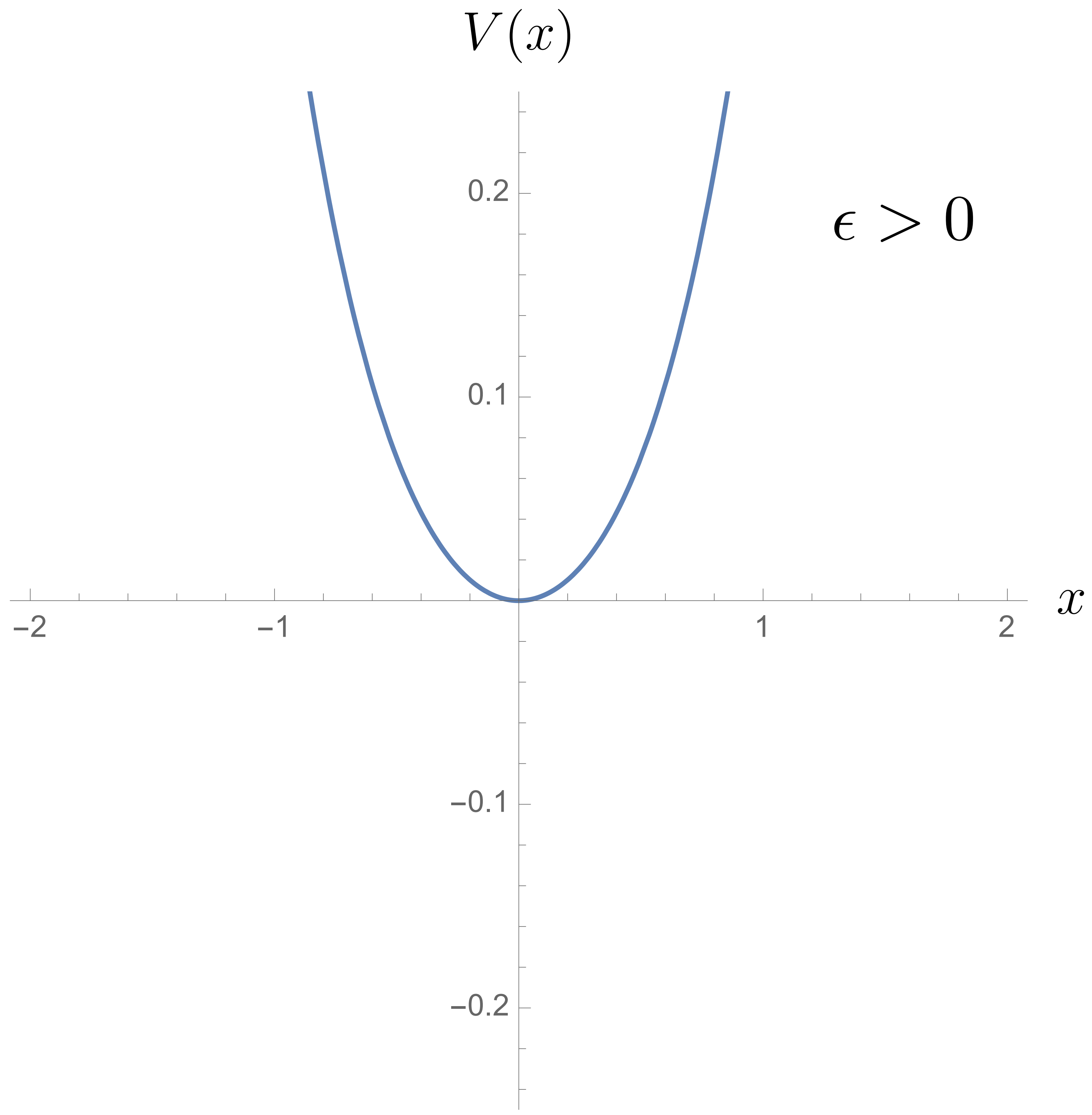
$E(\epsilon)$ must have a singularity at the origin in the complex ϵ plane

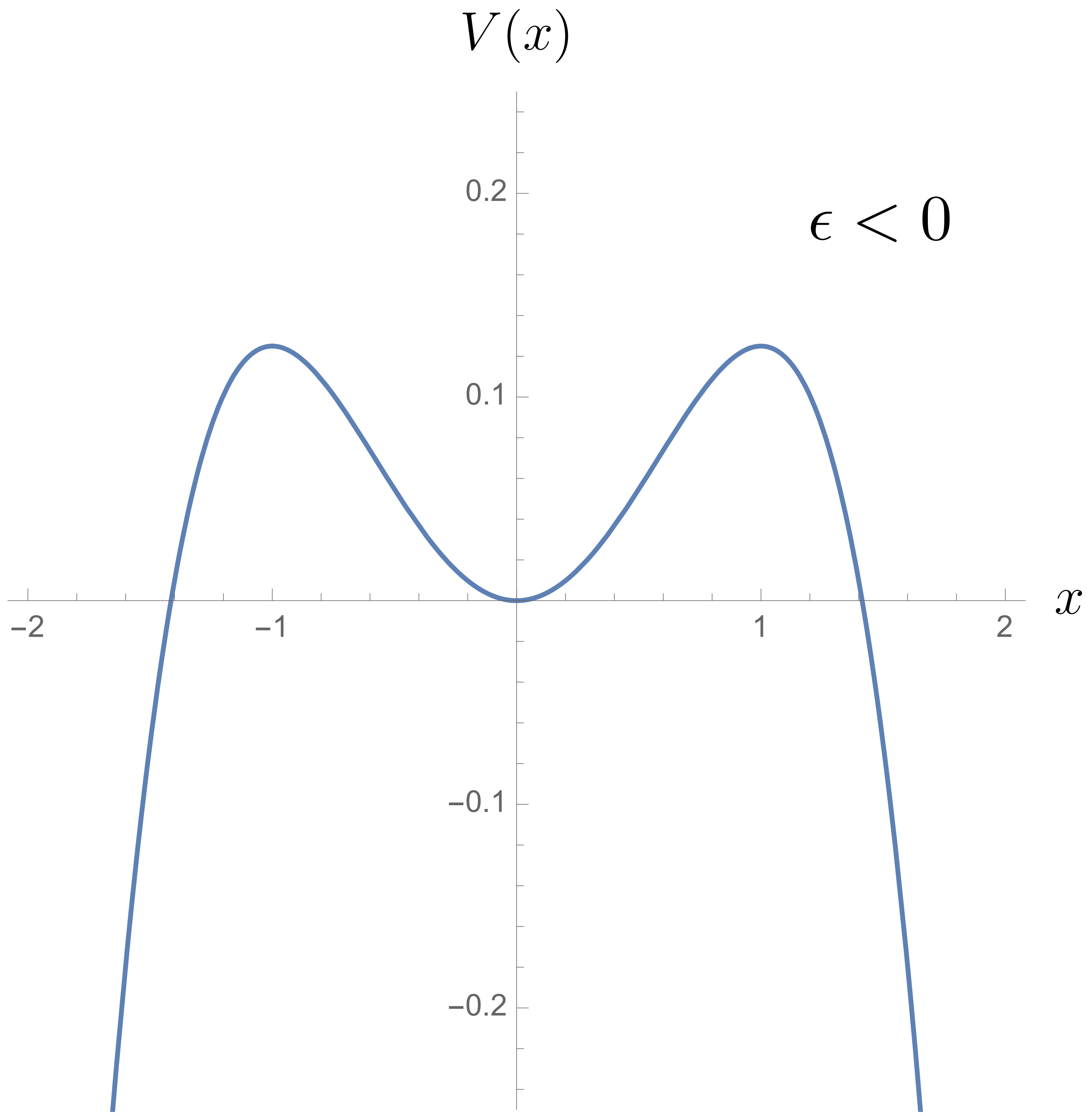




Origins of divergence

But **why** is there a singularity at the origin? $\Rightarrow$ Something abrupt happens at $\epsilon = 0$





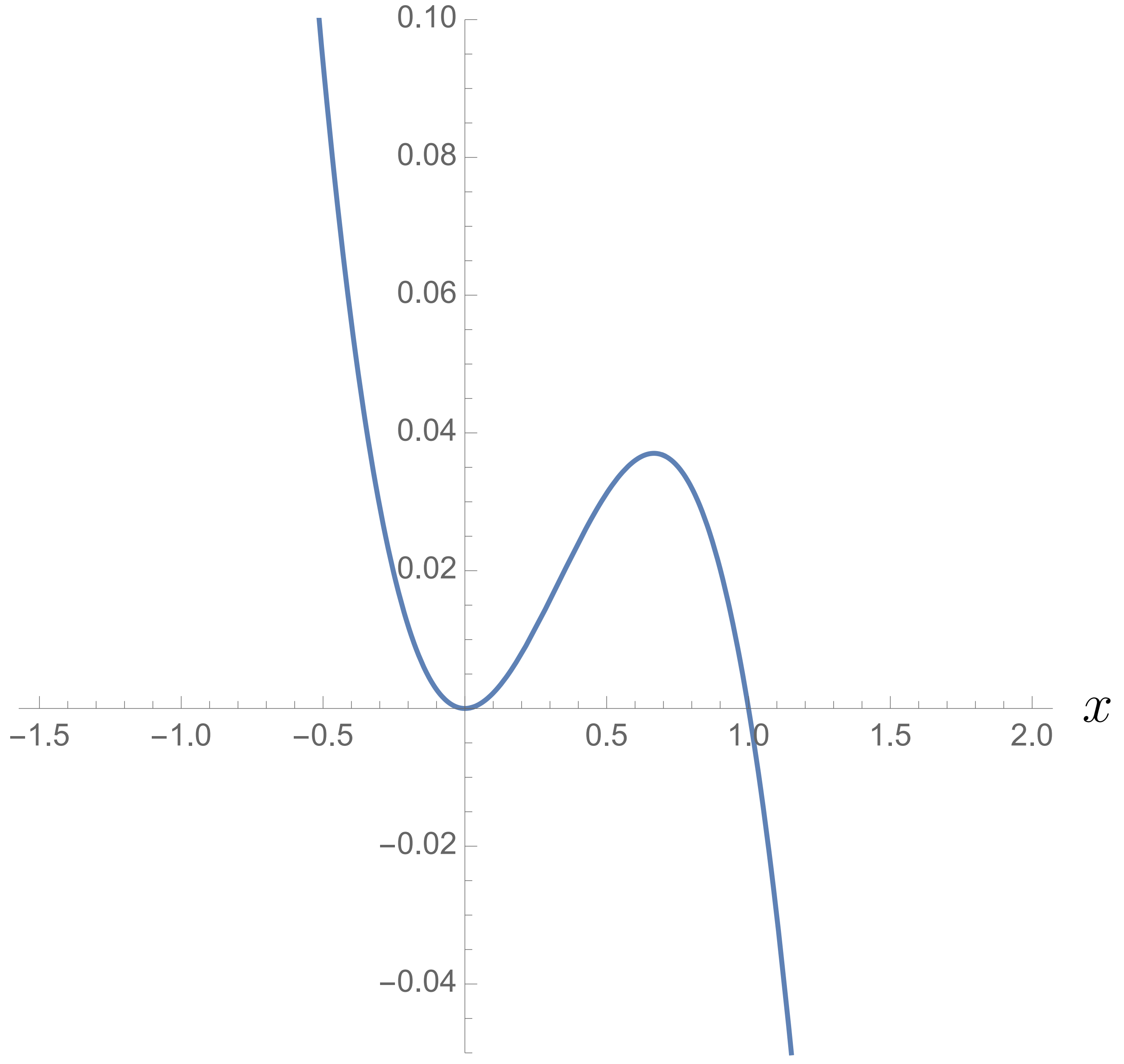
Complex energy

The divergence of perturbation theory signals a **physical instability** somewhere in the complex ϵ plane

$$\tilde{E} = E - \frac{i}{2}\hbar\Gamma \qquad \Longrightarrow \qquad P(t) = \left| e^{-i\tilde{E}t/\hbar} \right|^2 = e^{-\Gamma t}$$

"Inverse lifetime"

This is the generic behavior of perturbation theory in physics:



11.3 The Stark Effect in Hydrogen

$$E_1^{(1)} = e|\mathbf{E}|\langle 1, 0, 0|\hat{z}|1, 0, 0\rangle = 0 \qquad E_1^{(2)} = \sum_{n \neq 1, l, m} \frac{e^2|\mathbf{E}|^2 |\langle n, l, m|\hat{z}|1, 0, 0\rangle|^2}{E_1^{(0)} - E_n^{(0)}}$$

4.1 Perturbation Theory—Philosophy and Examples

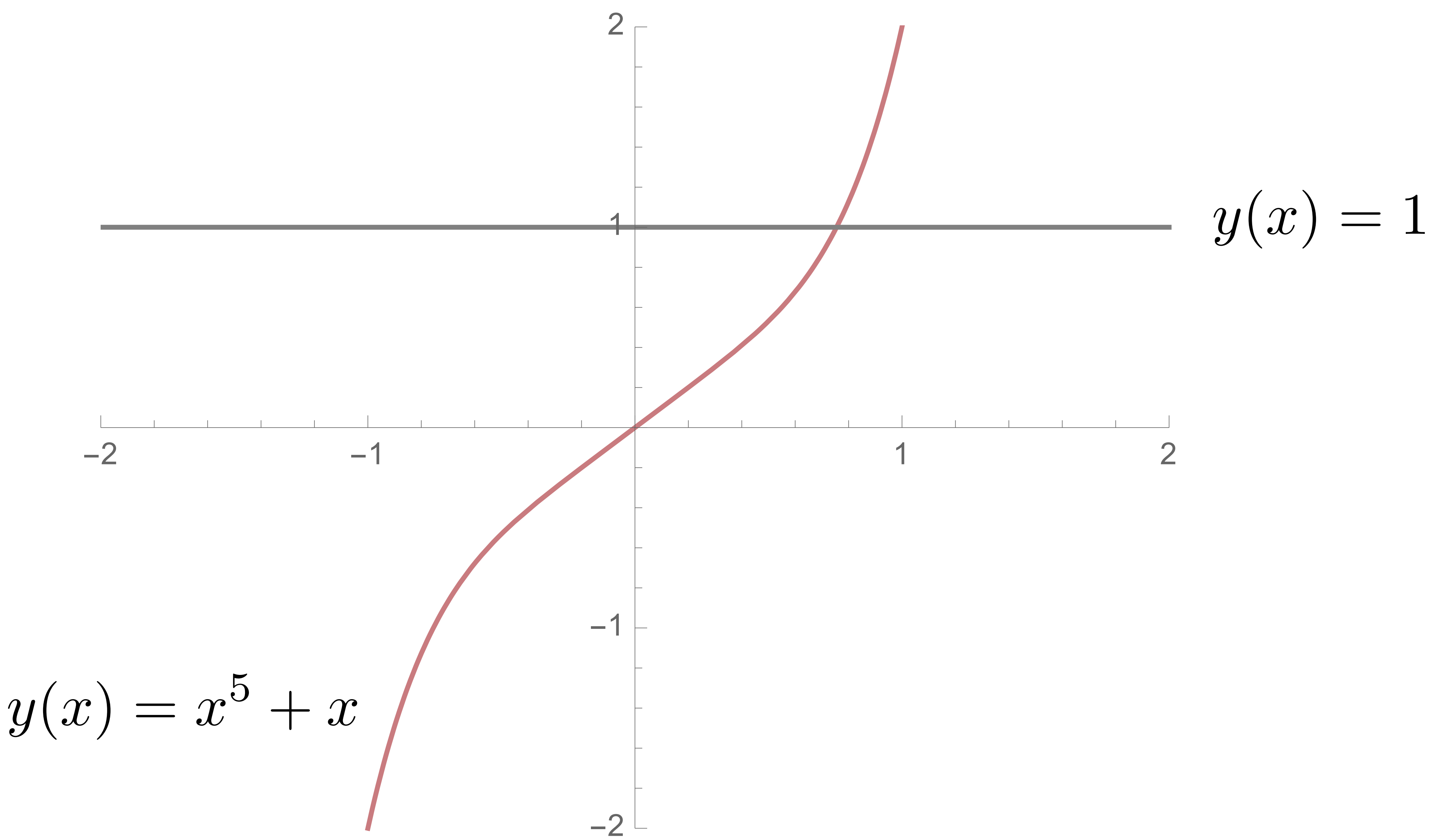
$$\langle\Omega| T\left[\phi(x_1)\cdots\phi(x_n)\right] |\Omega\rangle = \begin{pmatrix} \text{sum of all connected diagrams} \\ \text{with } n \text{ external points} \end{pmatrix}.$$

Practice

Consider the hard problem

$$x^5 + x = 1$$

Using perturbation theory, find the real root of this equation. Consider at least two different places to insert ϵ, do the answers agree? Why or why not?



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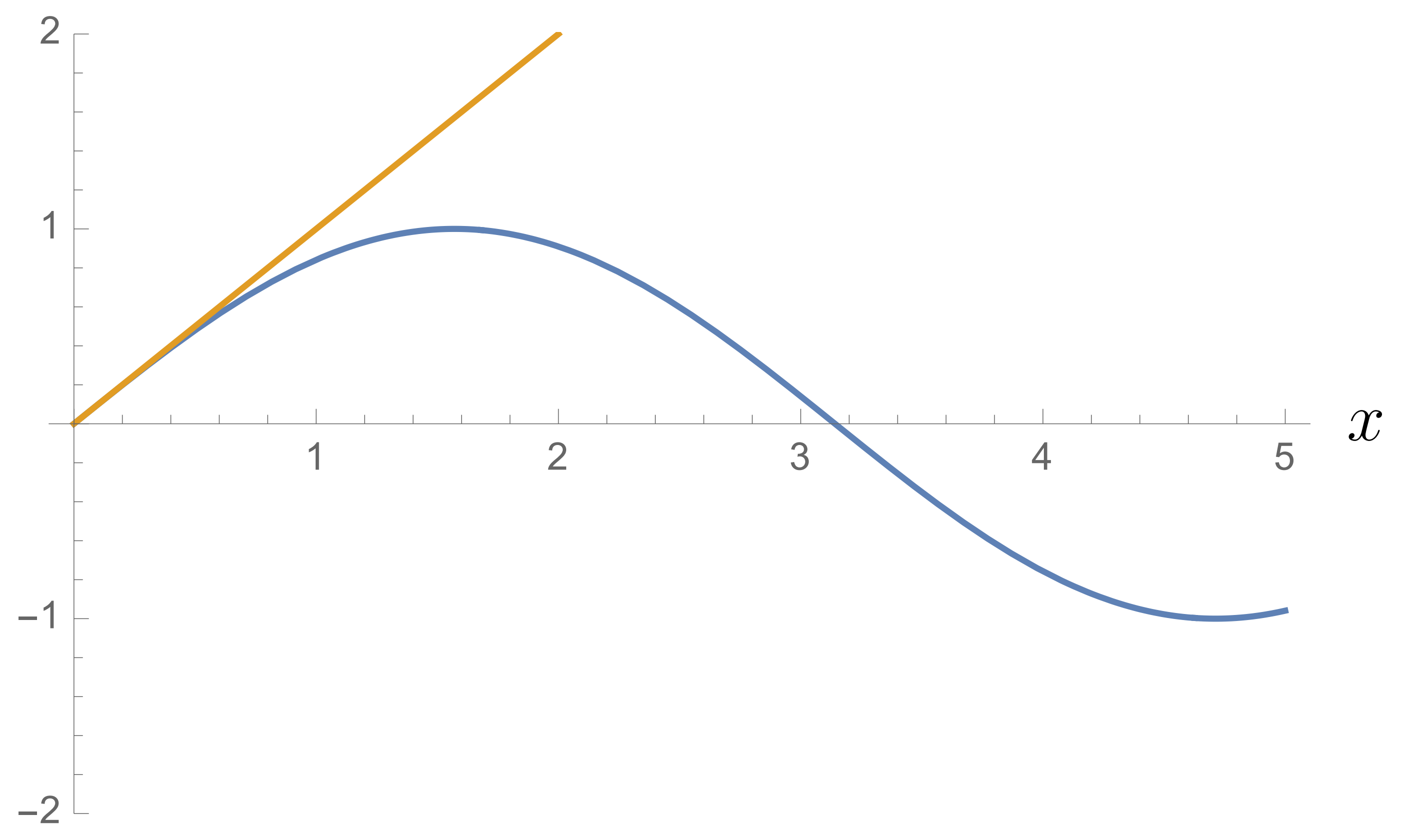
Asymptotics

Equalities are too rigid to handle divergent series

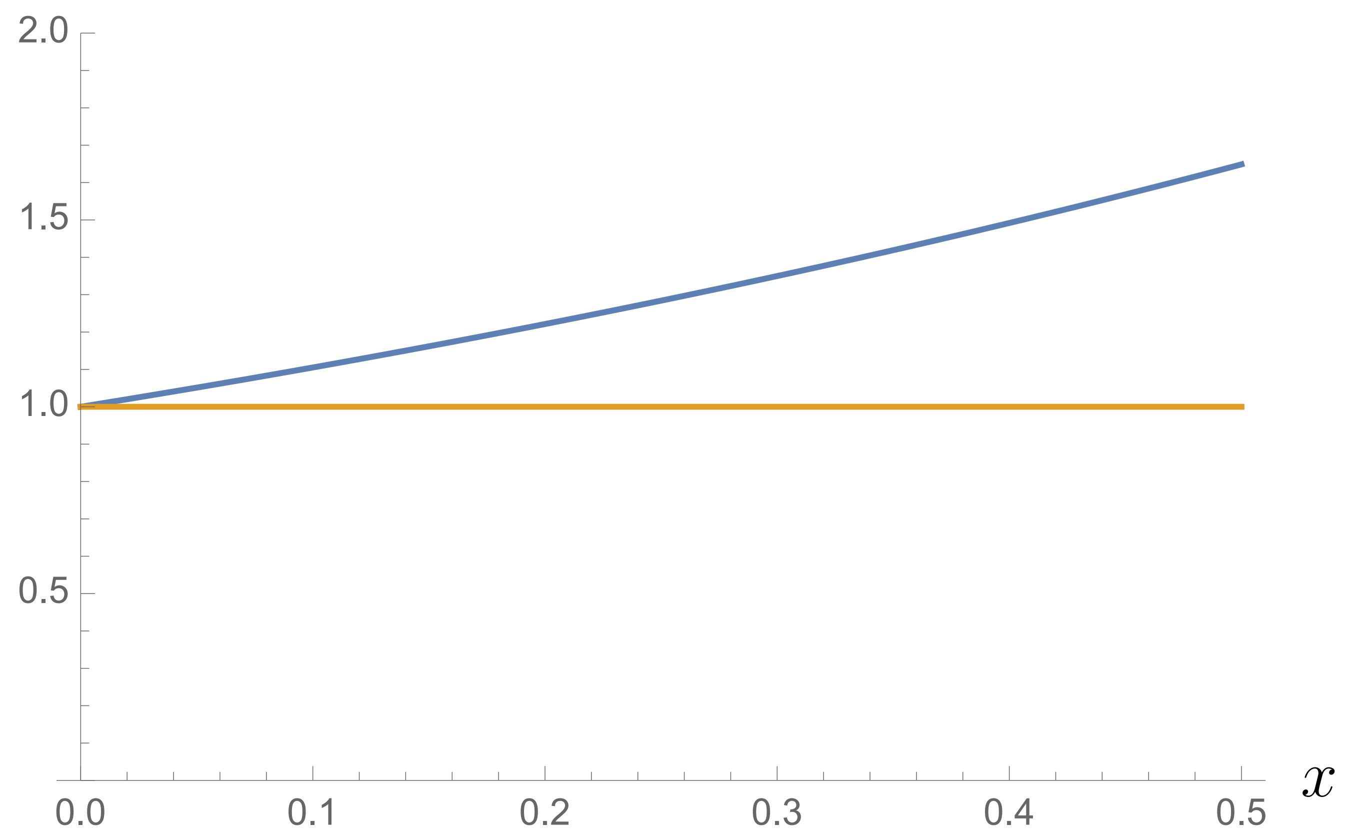
$$f(x) \sim g(x) \quad \text{as } x \to x_0 \qquad \Longleftrightarrow \qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

"Asymptotic relation"

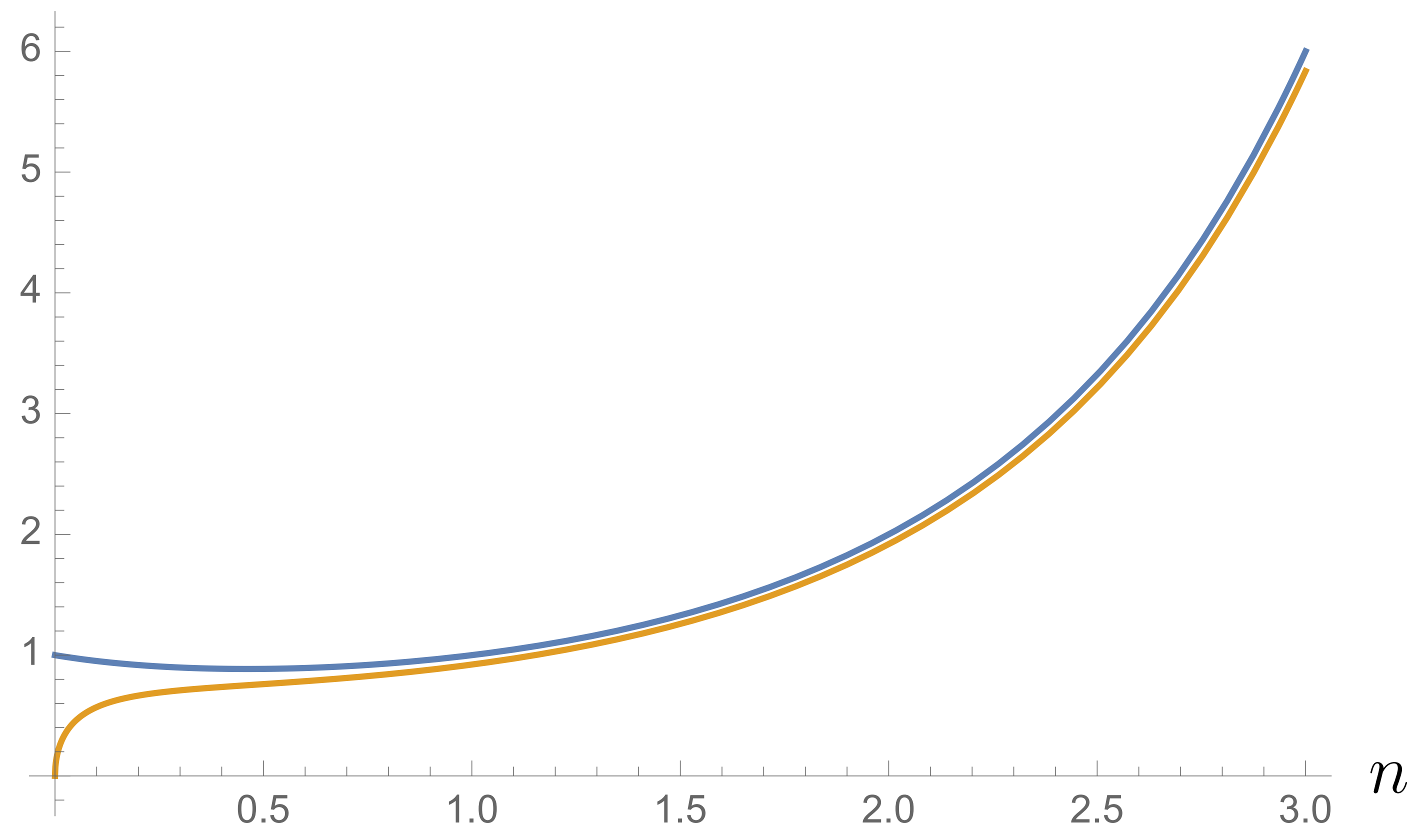
$$\sin x \sim x \quad \text{as } x \to 0$$



$$e^x \sim 1 \quad \text{as } x \to 0$$



$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \quad \text{as } n \to \infty$$

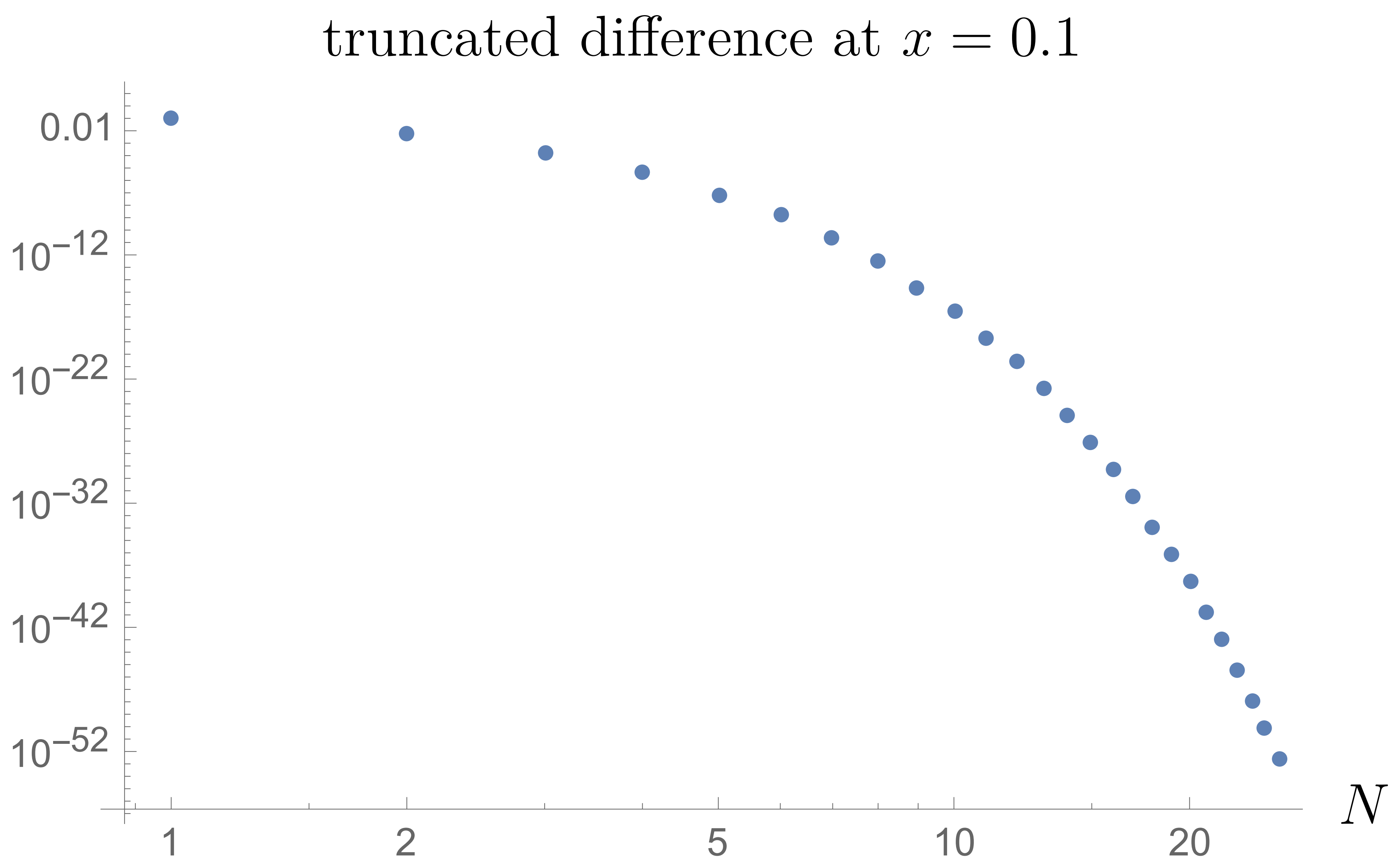


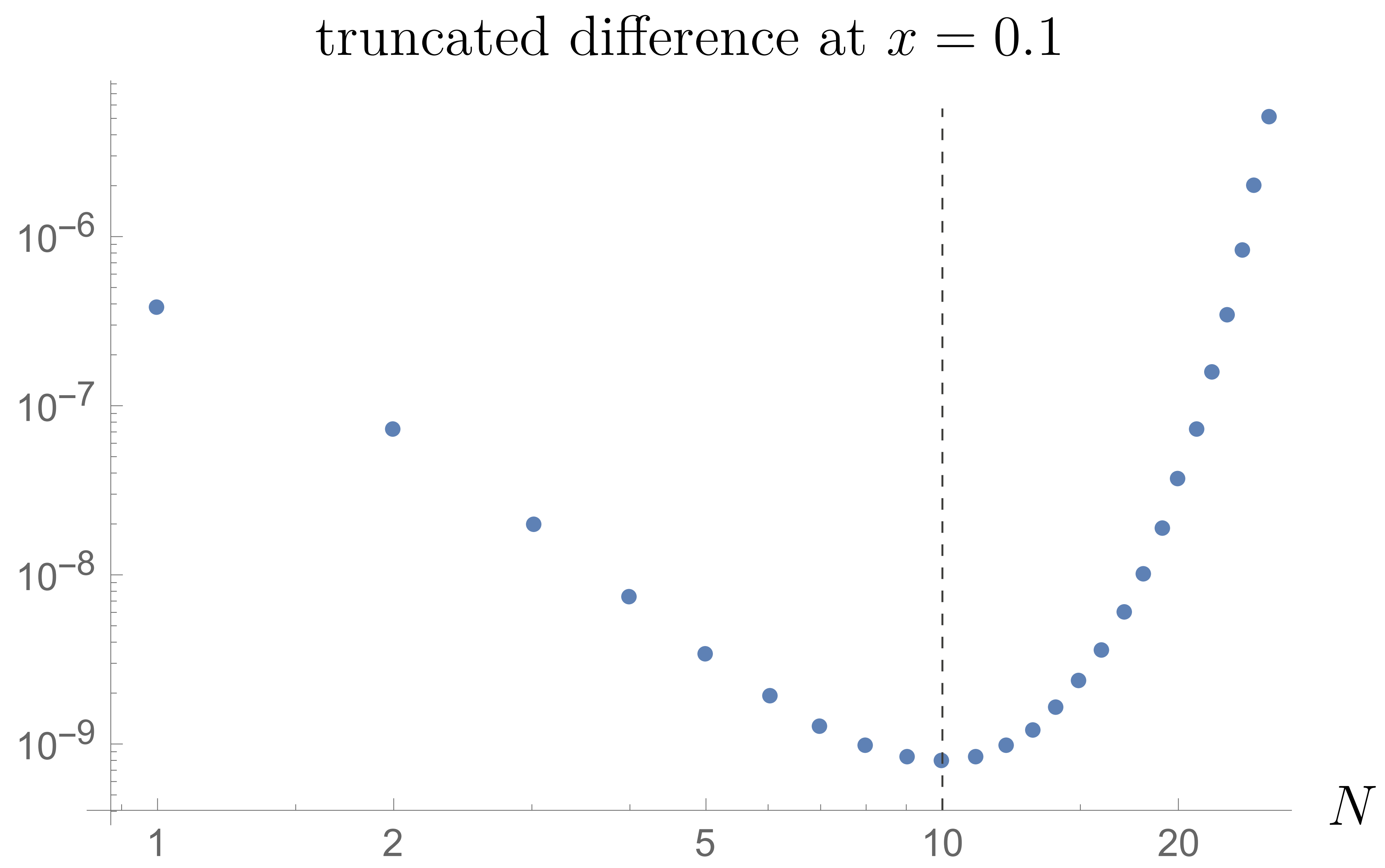
Functions can have **asymptotic expansions**: $f(x) \sim \sum_{n=0}^{\infty} a_n x^n \quad \text{as } x \to x_0$

Asymptotic vs. convergent

Exponential fn. : $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$

Incomplete gamma fn. : $\Gamma\left(0, \frac{1}{x}\right) \sim x e^{-1/x} \sum_{n=0}^{\infty} (-1)^n n! x^n$





At fixed x, asymptotic series approximate their parent function **until** $N^* \approx x^{-1}$ **"Optimal truncation"**

Successes of perturbation theory

Perturbation theory generally yields asymptotic expansions

$$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \qquad \text{as } \epsilon \to 0$$

e.g. electron's magnetic moment

$$\left.\frac{g}{2}\right|_{\text{th.}} = 1 + c_1\left(\frac{\alpha}{\pi}\right) + c_2\left(\frac{\alpha}{\pi}\right)^2 + c_3\left(\frac{\alpha}{\pi}\right)^3 + c_4\left(\frac{\alpha}{\pi}\right)^4 + c_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

$$\left.\frac{g}{2}\right|_{\text{th.}} = 1.001\,159\,652\,181\,78\,(6)(4)(3)(77)$$

$$\left.\frac{g}{2}\right|_{\text{exp.}} = 1.001\,159\,652\,180\,73\,(28)$$

Optimal truncation: $N^* = \left(\frac{\alpha}{\pi}\right)^{-1} \approx 430$

Borel summation

What do we want: a well-defined representation of $E(\epsilon)$ in the complex plane.

What do we have: a divergent, series representation on the real axis.

How to sum a divergent series: **Borel summation**

STEP 1: $f(x) \sim \sum_{n=0}^{\infty} c_n x^n \quad \text{as } x \to 0$

$c_n \sim n! \quad \text{as } n \to \infty$

$$\Longrightarrow \quad \mathcal{B}_f(t) := \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

"Borel transform"

STEP 2: $\mathcal{S}_f(x) := \int_0^{\infty} \mathrm{d}t\, e^{-t} \mathcal{B}_f(xt)$

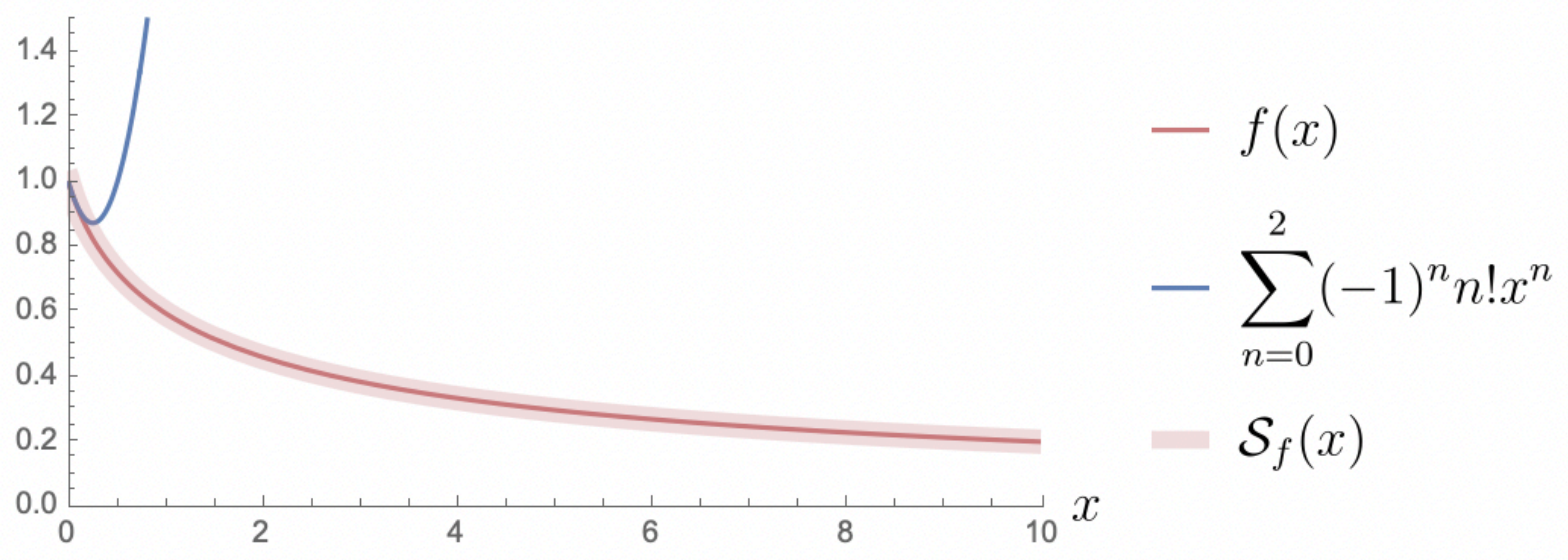
"Inverse Borel transform" or **"Borel sum"**

Example

$$f(x) = \frac{1}{x} e^{1/x} \Gamma\left(0, \frac{1}{x}\right) \sim \sum_{n=0}^{\infty} (-1)^n n! x^n \qquad \text{as } x \to 0^+$$

Borel transform: $\mathcal{B}_f(t) := \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t}$

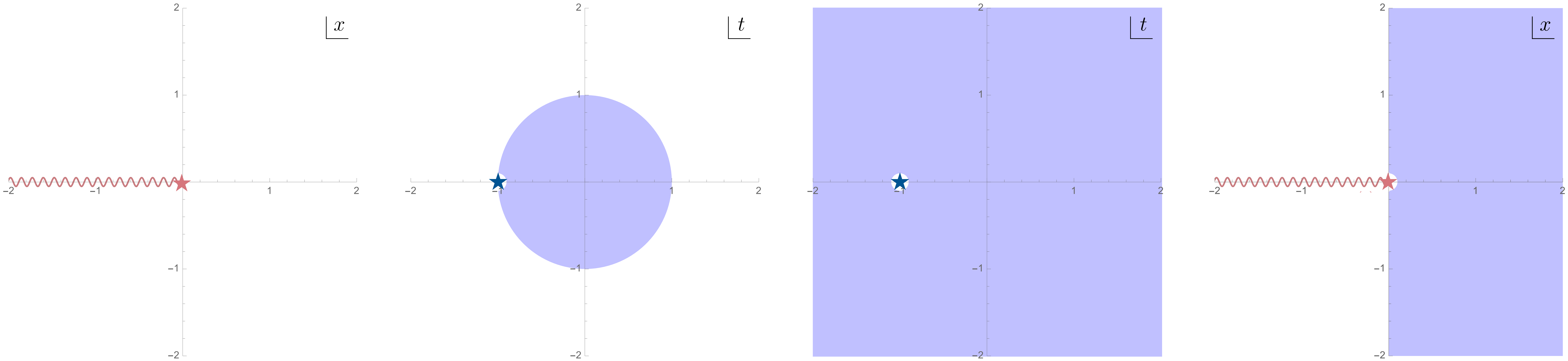
Borel sum: $\mathcal{S}_f(x) := \frac{1}{x} \int_0^{\infty} \mathrm{d}t \, e^{-t/x} \frac{1}{1+t}$ (compare with https://dlmf.nist.gov/8.6#E5)



How it works

All transforms must obey "conservation of information"

$$\sum_{n=0}^{\infty}(-1)^n n! x^n \implies \sum_{n=0}^{\infty}(-1)^n t^n \overset{!}{\implies} \frac{1}{1+t} \implies \mathcal{S}_f(x)$$



The power of Borel summation comes from **analytically continuing** the Borel transform

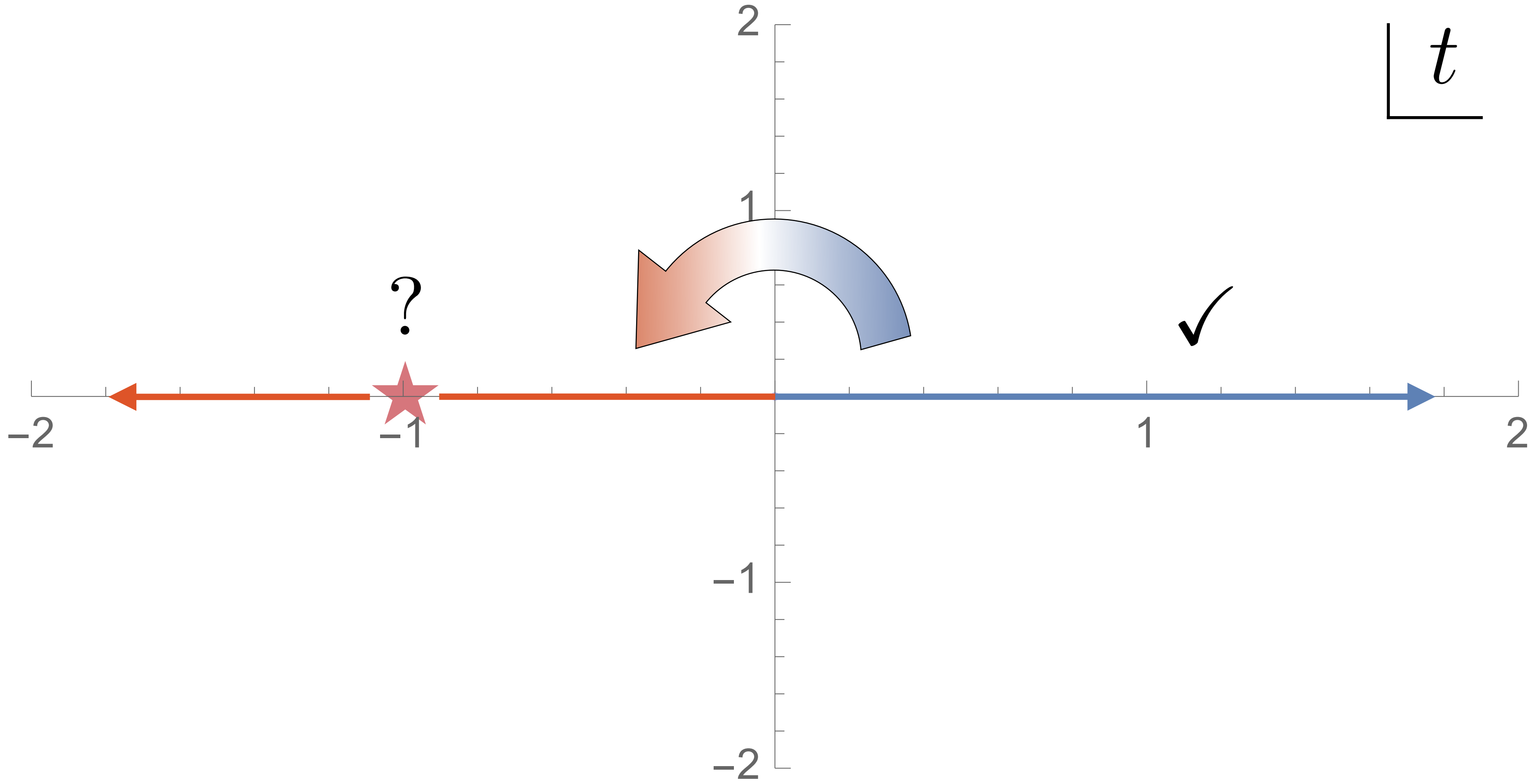
Singularities

What about $x < 0$?

$$S_f(-x) = -\frac{1}{x}\int_0^\infty \mathrm{d}t\, e^{t/x}\,\frac{1}{1+t} = \,?$$

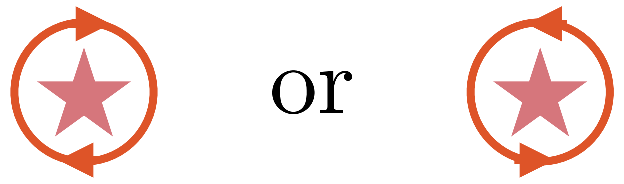
rotate contour

$$S_f(-x) = \frac{1}{x}\int_0^\infty \mathrm{d}t\, e^{-t/x}\,\frac{1}{1-t} = \,?$$



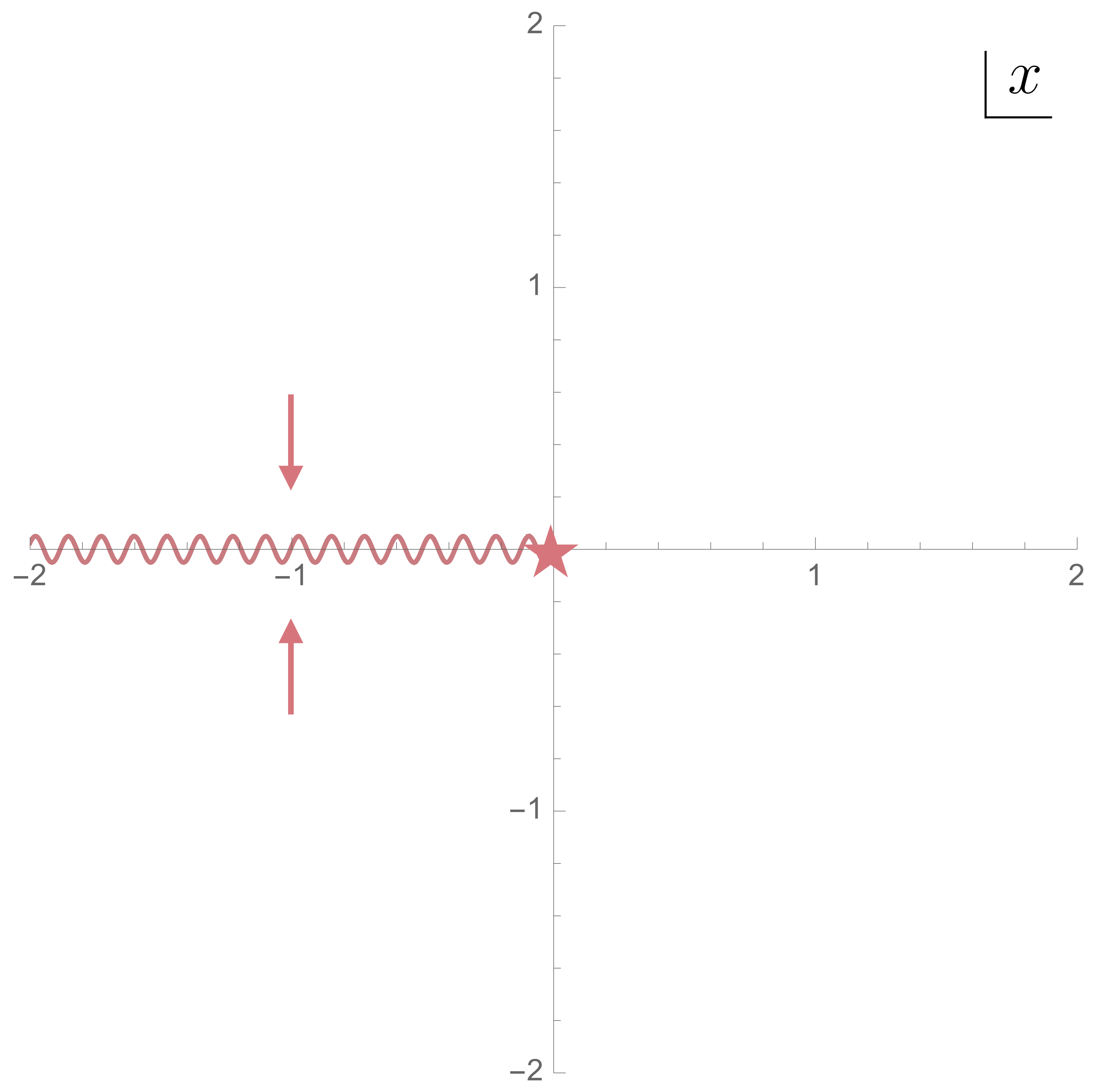
In order to avoid the singularity, we must deform the contour $\Rightarrow$ this introduces an **ambiguity**

$$S_f(-x) := \mathrm{P.V.}\left[\frac{1}{x}\int_0^\infty \mathrm{d}t\, e^{-t/x}\,\frac{1}{1-t}\right] \pm \pi i\left[\frac{1}{x}e^{-1/x}\right], \qquad S_f(xe^{i\pi}) - S_f(xe^{-i\pi}) = \pm 2\pi i\cdot\frac{1}{x}e^{-1/x}$$



Connection formula

The incomplete gamma function has a branch cut along the negative real axis



https://dlmf.nist.gov/8.2#E10 :

$$f\left(xe^{i\pi}\right) - f\left(xe^{-i\pi}\right) = 2\pi i \cdot \frac{1}{x} e^{-1/x}$$

"Jump across the cut" of the exact function

$$\Updownarrow$$

Residue of the singularity of the Borel transform!

Ambiguity in physical observables? $\implies$ Use physical knowledge of the system! $\tilde{E} = E - \frac{i}{2}\hbar\Gamma$

Take-away

Given an asymptotic expansion:

1. Determine the asymptotic behavior of the perturbative coefficients, typically $a_n \sim n! \qquad \text{as } n \to \infty$
2. Construct the Borel transform, analytically continue it, and study the singularities in the **Borel plane**.
3. Borel sum our original series and analytically continue it through the complex ϵ plane.
4. Use knowledge of the physical system to resolve imaginary ambiguities.

Real, **perturbative** expansion $\Longrightarrow$ Borel summation $\Longrightarrow$ **imaginary**, **non-perturbative** contributions

Physical observables are represented by **trans-series**

$$E(\epsilon) \sim \sum_{n=0}^{\infty} c_n \epsilon^n \qquad \Longrightarrow \qquad E(\epsilon) \stackrel{!}{=} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell-1} c_{n\ell k}\, \epsilon^n \left[\exp\left(-\frac{\mathcal{A}}{\epsilon}\right)\right]^{\ell} \left[\ln\left(-\frac{1}{\epsilon}\right)\right]^{k}$$

Practice

The Airy function has an asymptotic expansion

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right) \sum_{n=0}^{\infty} \frac{a_n}{x^{3n/2}} \qquad \text{as } x \to \infty$$

where the coefficients have the asymptotic form

$$a_n = (-1)^n \frac{\Gamma\left(n+\frac{1}{6}\right)\Gamma\left(n+\frac{5}{6}\right)}{2\pi n!\left(\frac{4}{3}\right)^n} \sim \frac{1}{2\pi}(-1)^n \Gamma(n) \left(\frac{3}{4}\right)^n \qquad \text{as } n \to \infty$$

Use Borel summation to:

1. Define the Borel transform and identify the singularity structure
2. Define the Borel sum of the series and compare with the exact Airy function
3. Analytically continue the Borel sum near the singularity and compare with the Airy function connection formula (https://dlmf.nist.gov/9.2#E12) (hint: use https://dlmf.nist.gov/15.2#E3)

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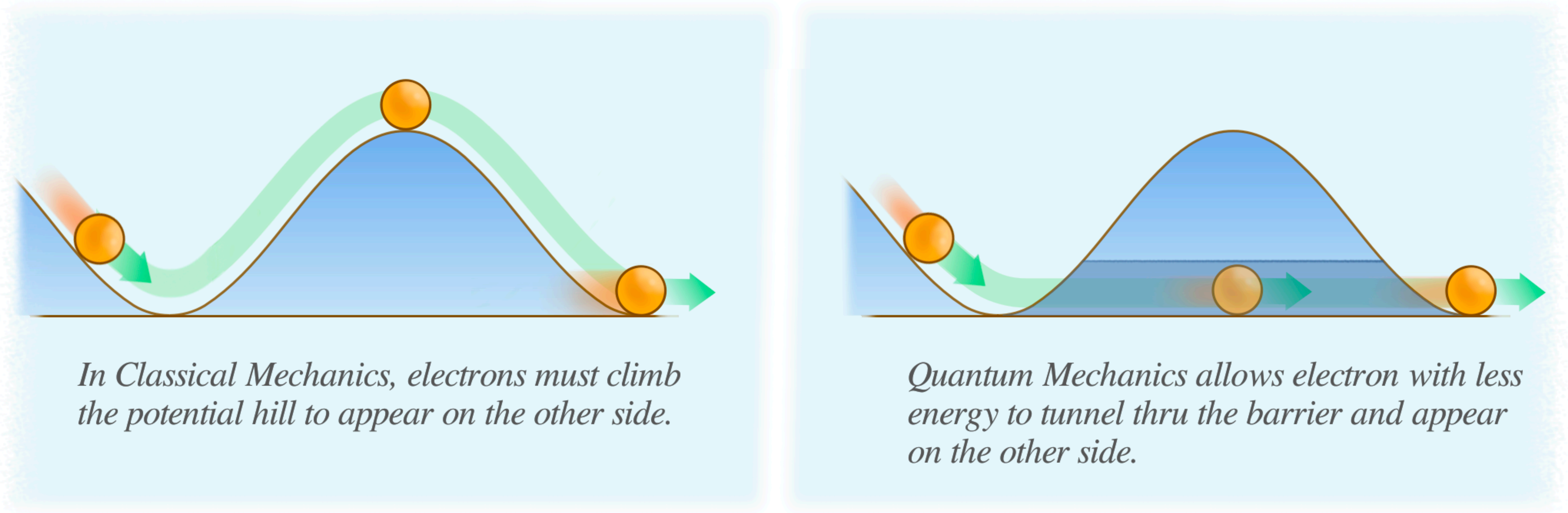
Tunneling

The most famous non-perturbative phenomenon is **tunneling**



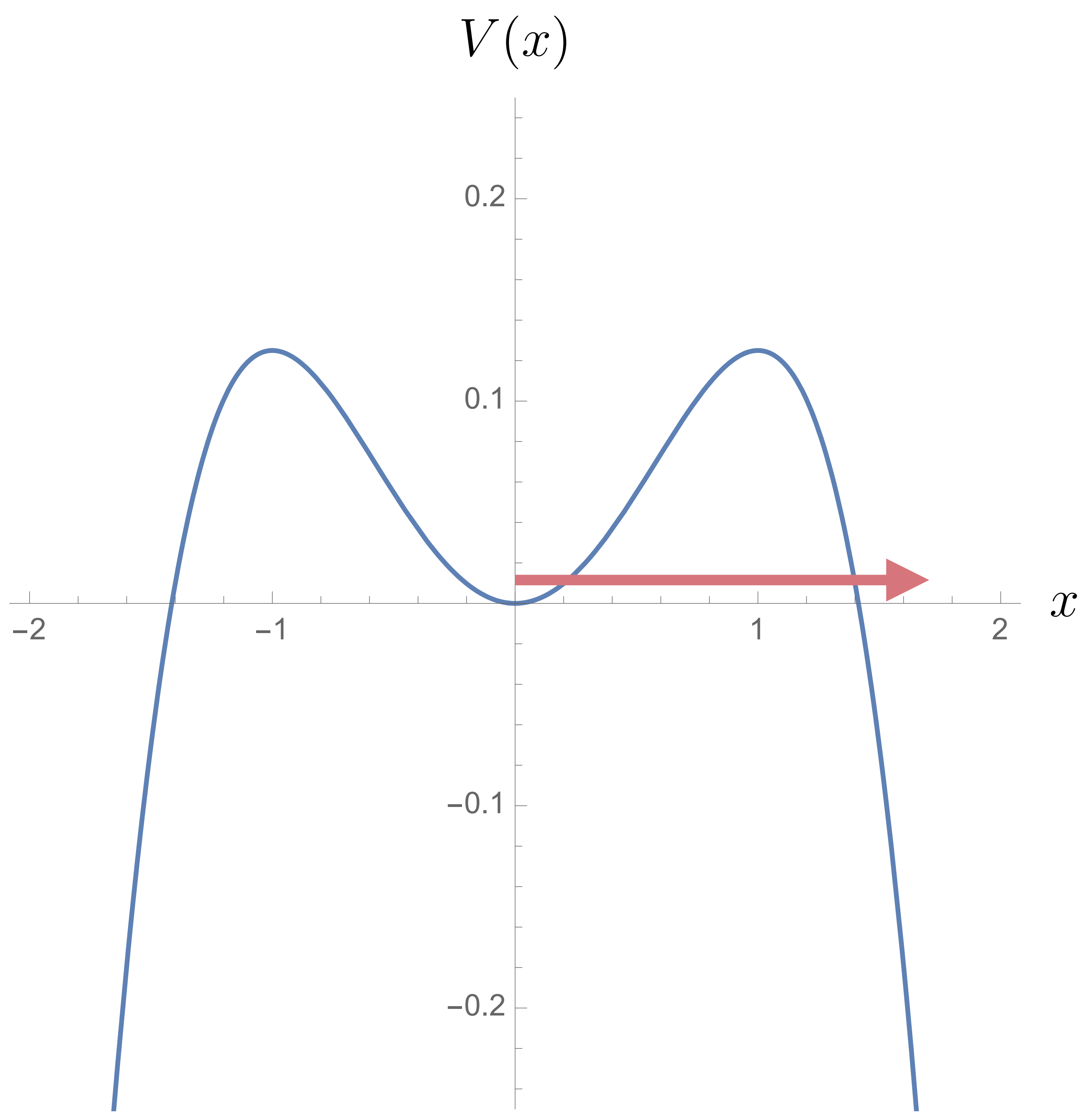
The **transmission coefficient** gives the probability for tunneling, and is **non-perturbative** in the barrier height

$$V(x) = \frac{x^2}{4} - \epsilon \frac{x^4}{4} \qquad \Longrightarrow \qquad T \approx e^{-1/\epsilon}$$

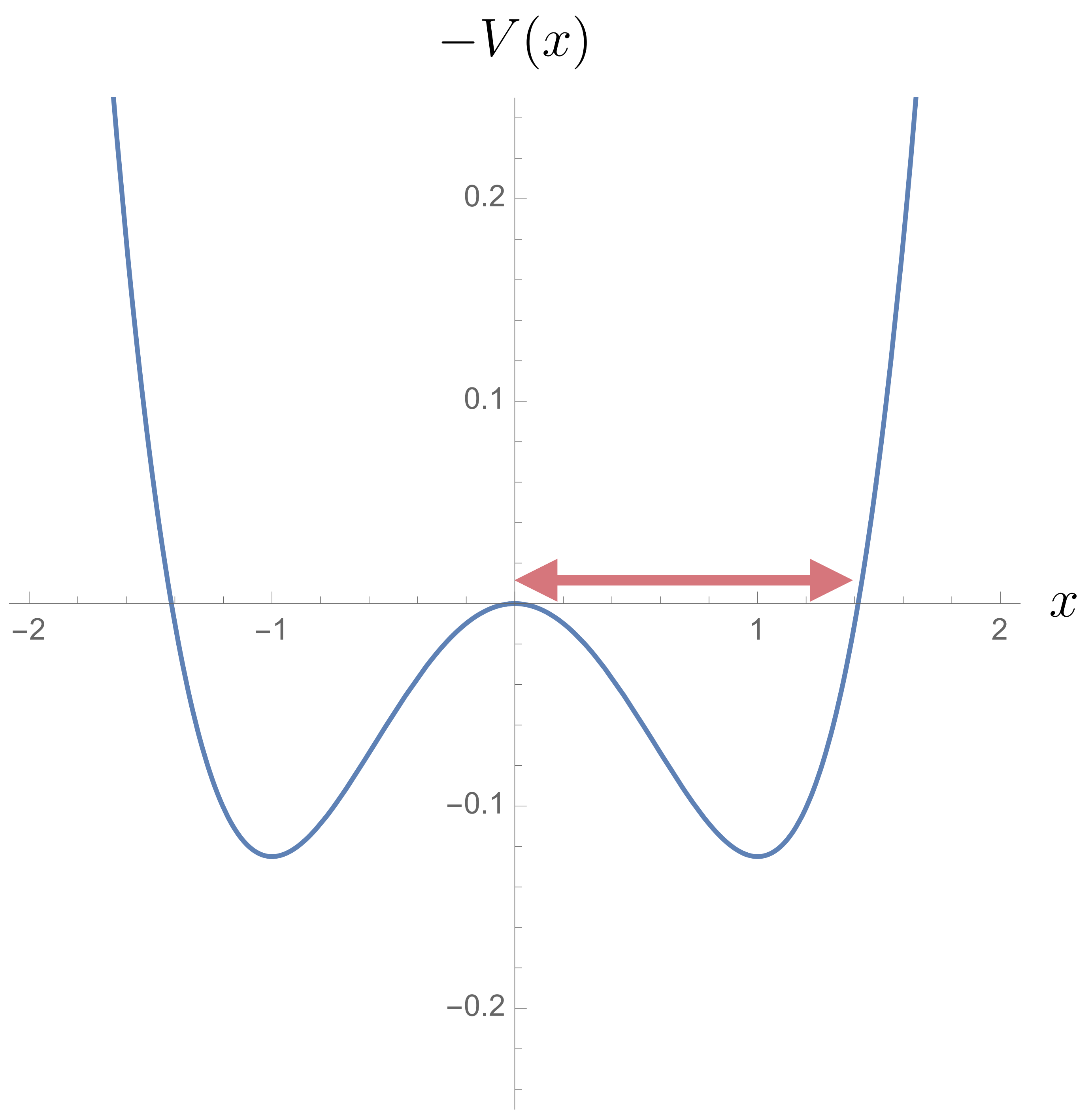
Methods of calculation

What techniques do we have for calculating tunneling rates?

WKB approximation



Path integral



Perturbation theory

$$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \qquad \text{as } \epsilon \to 0$$

$$a_n \sim n! \qquad \text{as } n \to \infty$$

?

WKB

Free particle solution to Schrödinger equation

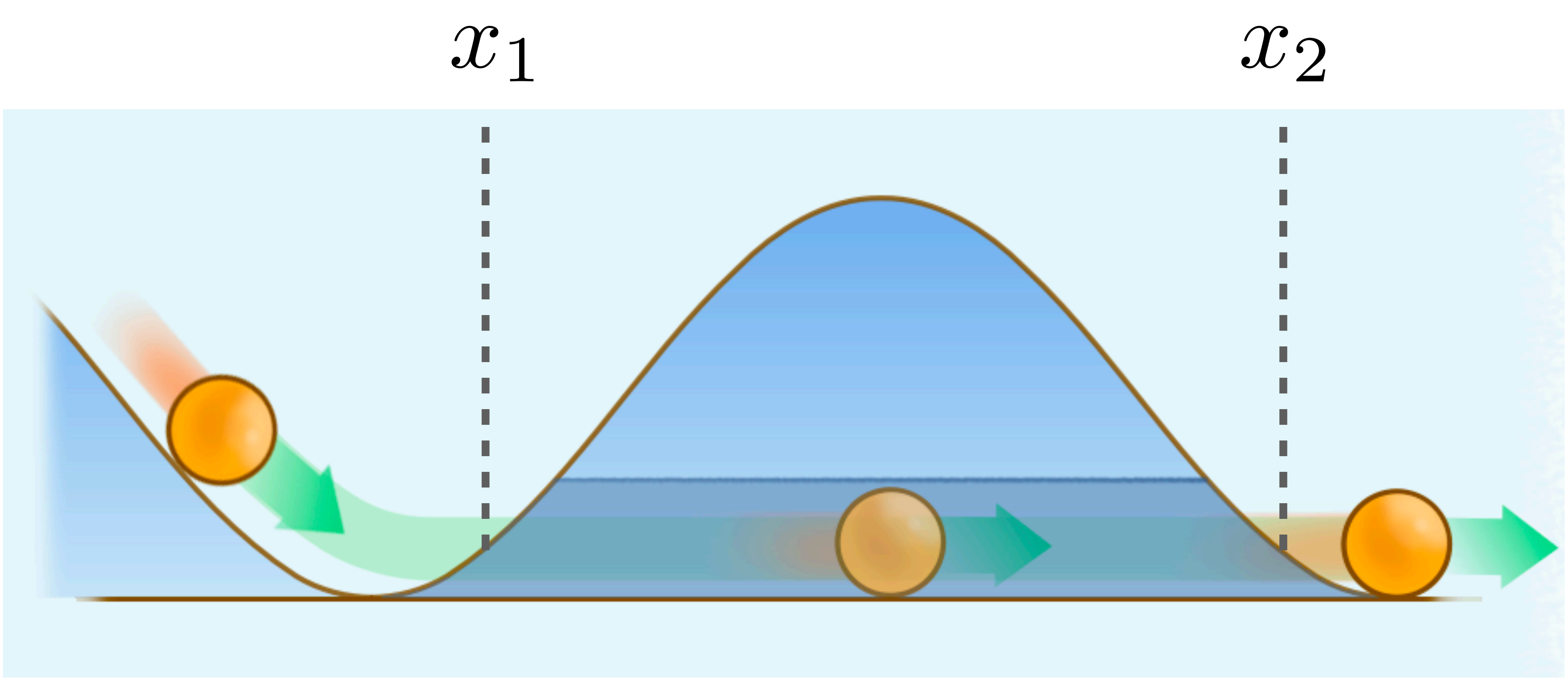
$$H\psi(x) = E\psi(x), \qquad H = \frac{p^2}{2m} \qquad \Longrightarrow \qquad \psi(x) = \exp\left(\pm\frac{i}{\hbar}px\right)$$

The **WKB approximation** modifies this result for slowly varying potentials

$$\psi(x) \approx \exp\left(\pm\frac{i}{\hbar}\int^x \mathrm{d}z\, p(z)\right), \qquad p(x) = \sqrt{2m(E - V(x))}$$

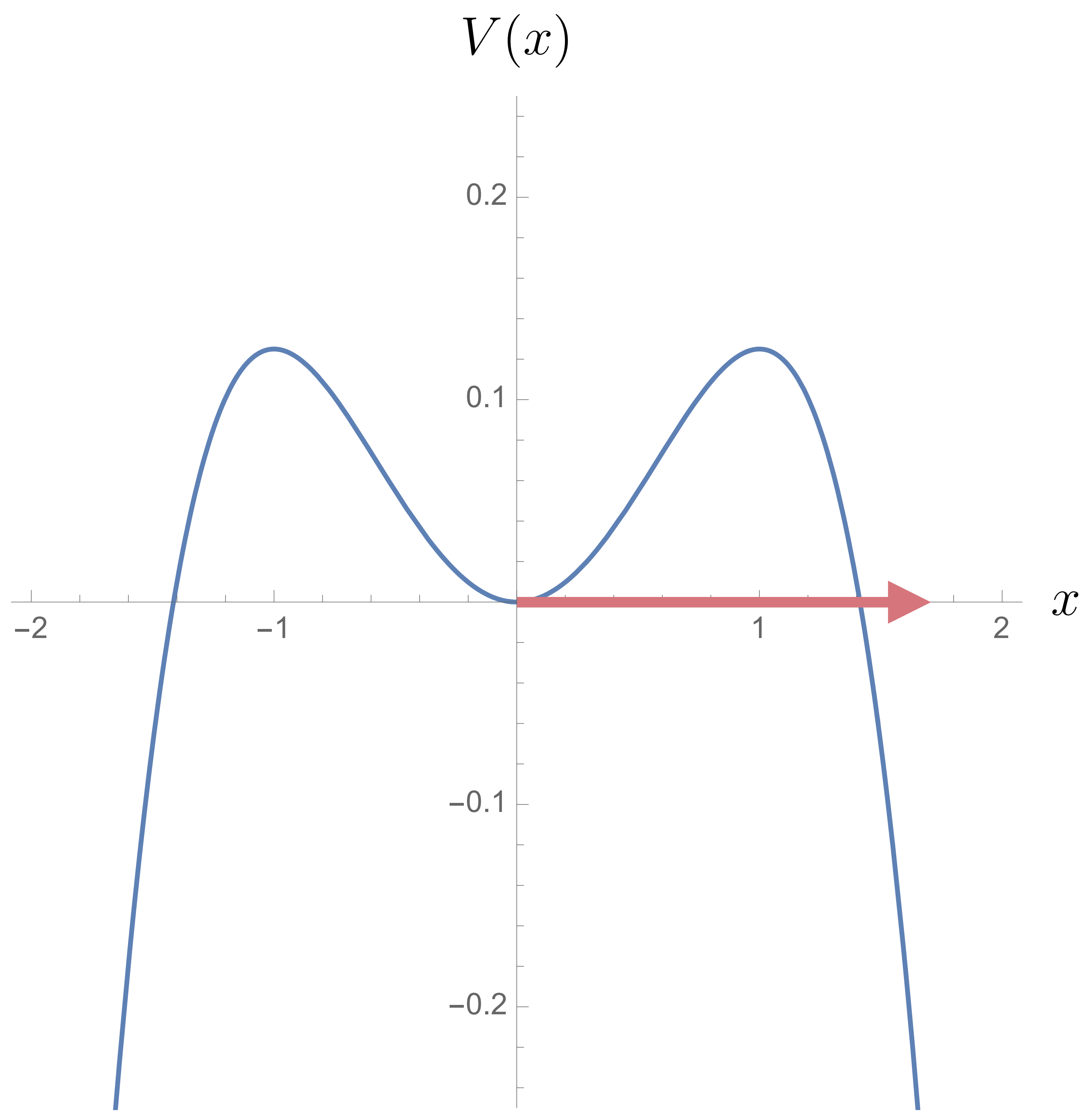
Transmission coefficient given by:

$$T \approx \exp\left(-\frac{2}{\hbar}\int_{x_1}^{x_2} \mathrm{d}x\, |p(x)|\right)$$



Anharmonic oscillator - WKB

Let's assume $E \approx 0$ to make the turning points easy:



$$V(x) = \frac{x^2}{4} - \epsilon\frac{x^4}{4} \qquad \Rightarrow \qquad x_1 = 0, \quad x_2 = \frac{1}{\sqrt{\epsilon}}$$

$$T \approx \exp\left(-2\int_0^{1/\sqrt{\epsilon}} \mathrm{d}x \sqrt{\frac{x^2}{4} - \frac{x^4}{4}}\right)$$

$$= \exp\left(-2 \cdot \frac{1}{6\epsilon}\right)$$

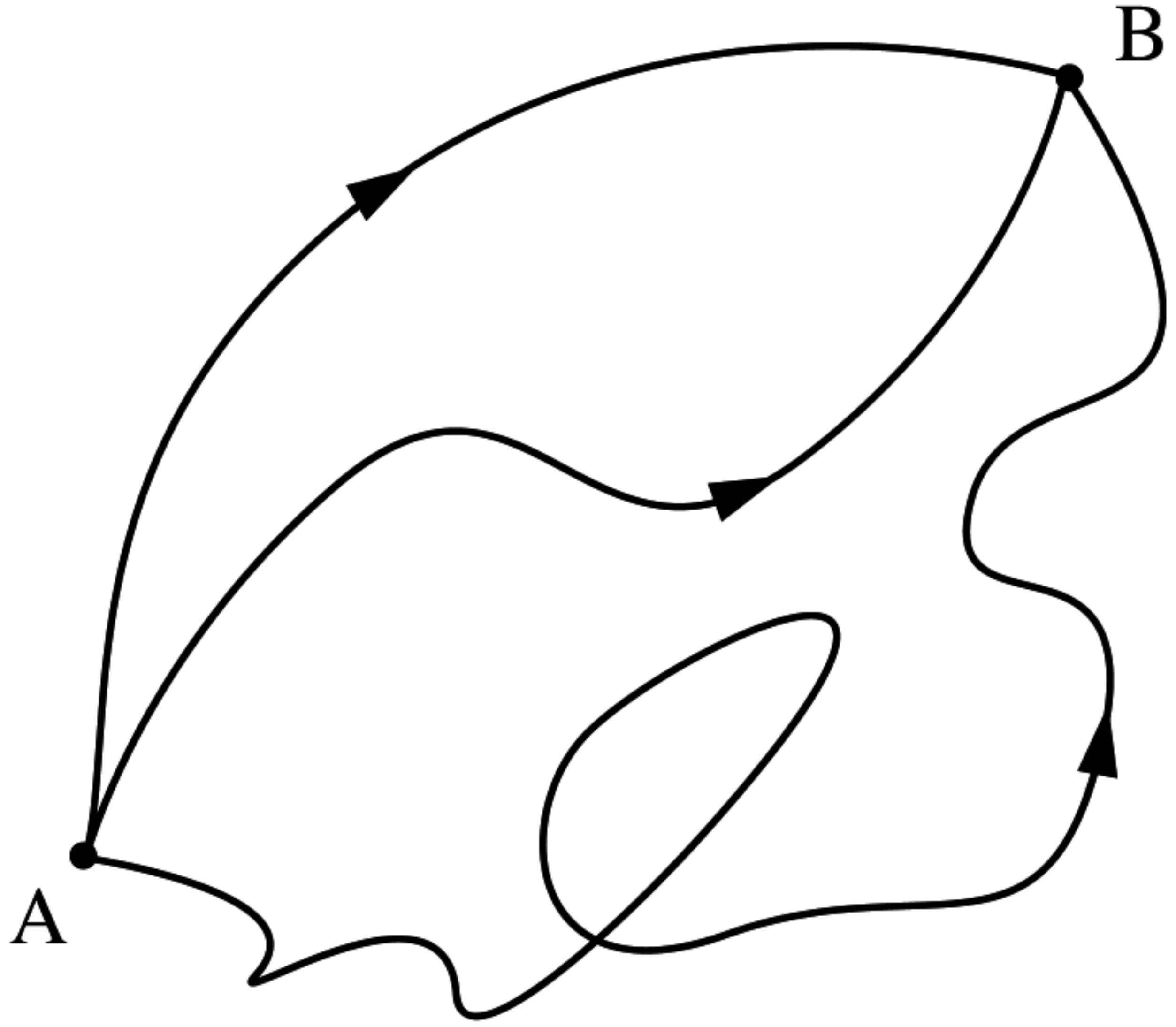
$$T \approx \exp\left(-\frac{1}{3\epsilon}\right)$$

As expected, the transmission coefficient is **non-perturbative** in the barrier height.

Path integral

Feynman **path integral** gives amplitudes as "sums over all possible paths"

$$\langle B, t_2 | A, t_1 \rangle \quad = $$



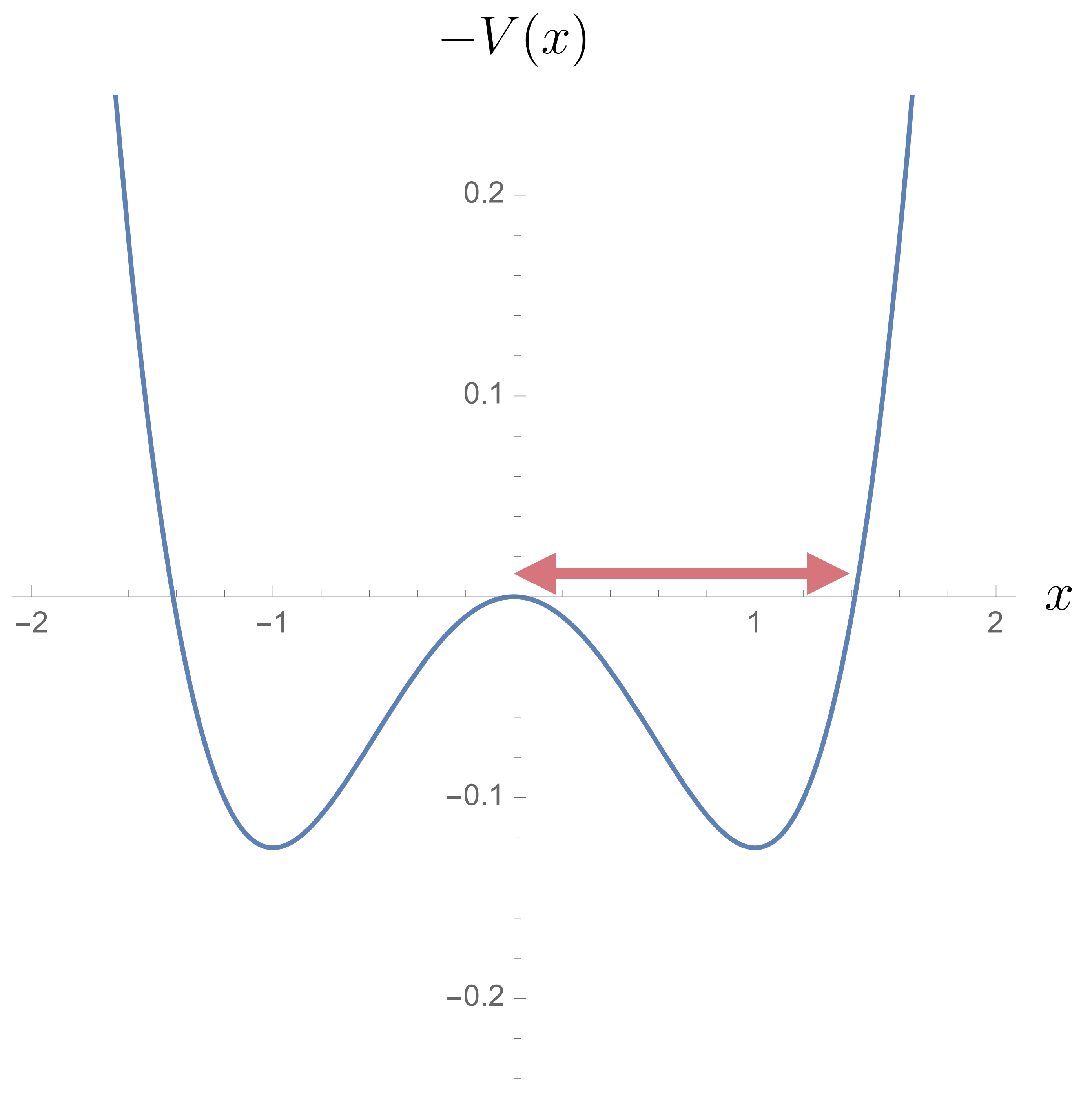
$$= \quad \int \mathcal{D}x \, \exp\left(\frac{i}{\hbar} S[x(t)]\right)$$

Each path is weighted by a phase factor depending on the **classical action**

$$S[x(t)] = \int \mathrm{d}t \left(\frac{1}{2} m \dot{x}(t)^2 - V(x(t))\right)$$

Anharmonic oscillator - instantons

Tunneling processes are described by classical motion in an **inverted potential**



These classical solutions are called **instantons**

$$\dot{x}(t) = \pm\sqrt{\frac{2}{m}V(x(t))}$$

$$\dot{x}(t) = \pm 2\sqrt{\frac{1}{4}x(t)^2 - \frac{\epsilon}{4}x(t)^4}$$

$$x(t) = \frac{1}{\sqrt{\epsilon}}\operatorname{sech} t$$

$$T \approx \exp\left(-\frac{1}{\hbar}S[x(t)]\right) = \exp\left(-\int_{-\infty}^{\infty} \mathrm{d}t \left[\frac{1}{4\epsilon}\operatorname{sech}^2(t)\tanh^2(t) + \frac{1}{4\epsilon}\operatorname{sech}^2(t) - \frac{1}{4\epsilon}\operatorname{sech}^4(t)\right]\right) = \exp\left(-\frac{1}{3\epsilon}\right)$$

Large-order growth

This information is contained within our perturbative expansion

$$E(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n \quad \Longrightarrow \quad \mathcal{B}(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n \quad \Longrightarrow \quad \mathcal{S}_E(\epsilon) = \frac{1}{\epsilon} \int_0^{\infty} \mathrm{d}t \, e^{-t/\epsilon} \mathcal{B}(t)$$

Large-order growth: $a_n \sim (-1)^n \beta^n \Gamma(\mu n + \nu) \qquad$ as $n \to \infty$

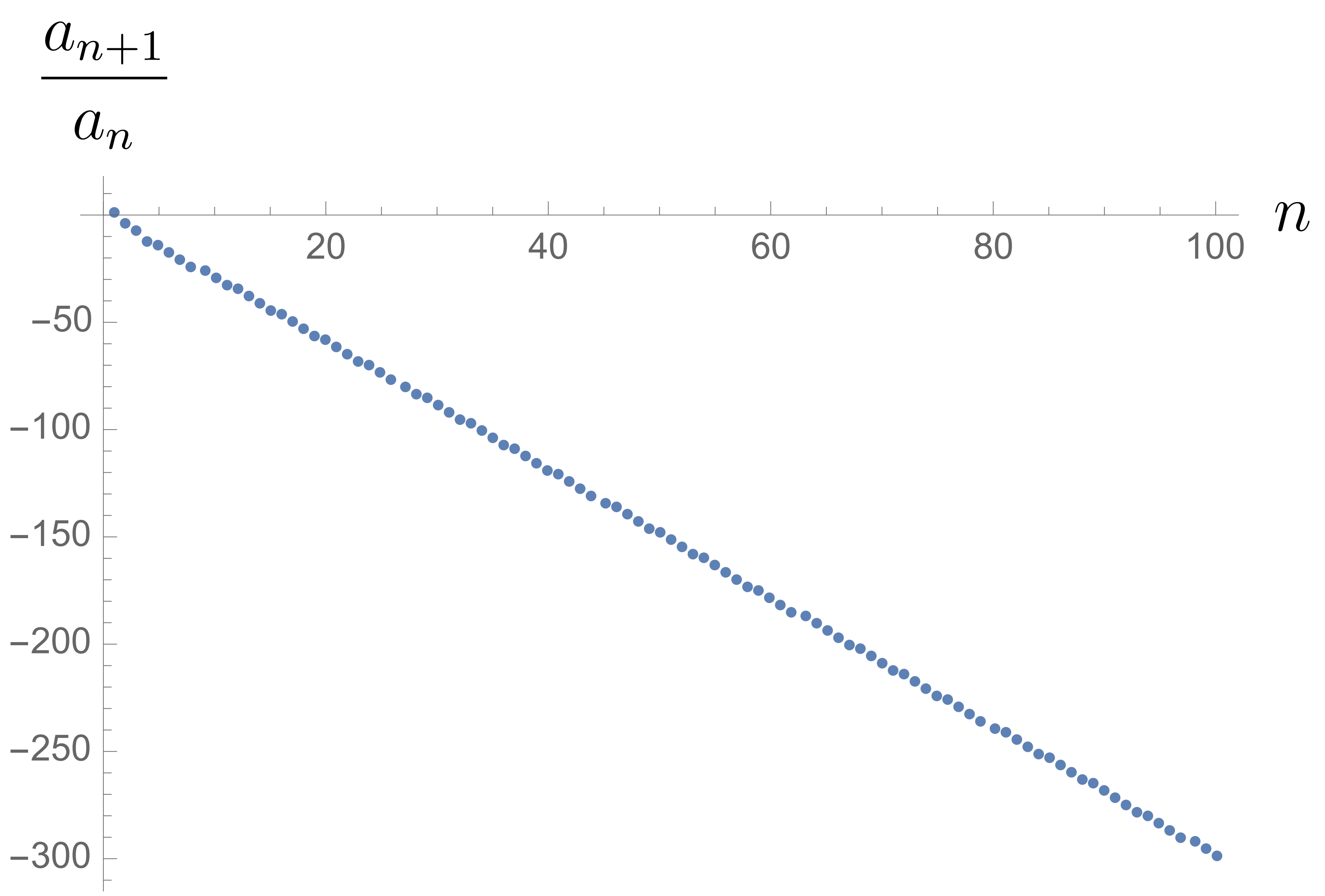
Step 1: plot the ratio $\dfrac{a_{n+1}}{a_n} \sim -\beta(\mu n)^{\mu} \qquad$ as $n \to \infty$

Step 2: plot the ratio $\dfrac{a_n}{(-1)^n \beta^n \Gamma(\mu n + \nu)} \sim$ constant for the correct choice of $\nu \qquad$ as $n \to \infty$

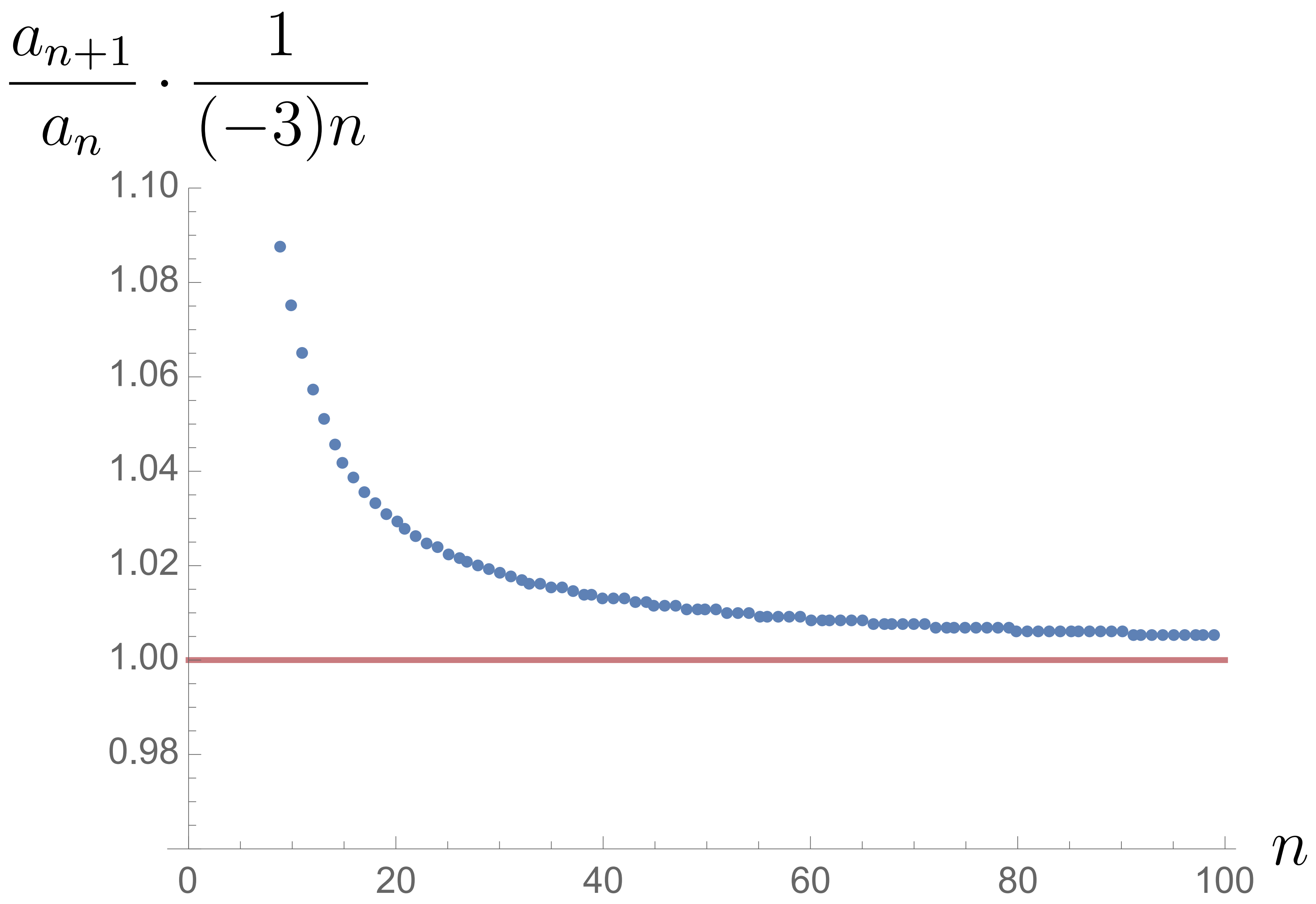
Nearest singularity in Borel plane: $|t| = \beta^{-1}$

Anharmonic oscillator - PT

$$a_n = \left\{ \frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16}, -\frac{30885}{128}, \frac{916731}{256}, -\frac{65518401}{1024}, \frac{2723294673}{2048}, -\frac{1030495099053}{32768}, \frac{54626982511455}{65536}, \ldots \right\}$$



$\Longrightarrow$



The large-order growth encodes the non-perturbative physics!

$$\mathcal{S}_E(\epsilon) = \frac{1}{\epsilon} \int_0^\infty \mathrm{d}t \, e^{-t/\epsilon} \mathcal{B}(t) \qquad \Longrightarrow \qquad \mathrm{Im}\, \mathcal{S}_E(-\epsilon) \approx \exp\left(-\frac{1}{\beta\epsilon}\right) = \exp\left(-\frac{1}{3\epsilon}\right)$$

Finite-order resummation

What if we only know the first N terms?

$$E(\epsilon) \sim \sum_{n=0}^{N-1} a_n \epsilon^n \qquad \text{as } \epsilon \to 0 \qquad \Longrightarrow \qquad \mathcal{B}(t) = \sum_{n=0}^{N-1} \frac{a_n}{n!} t^n$$

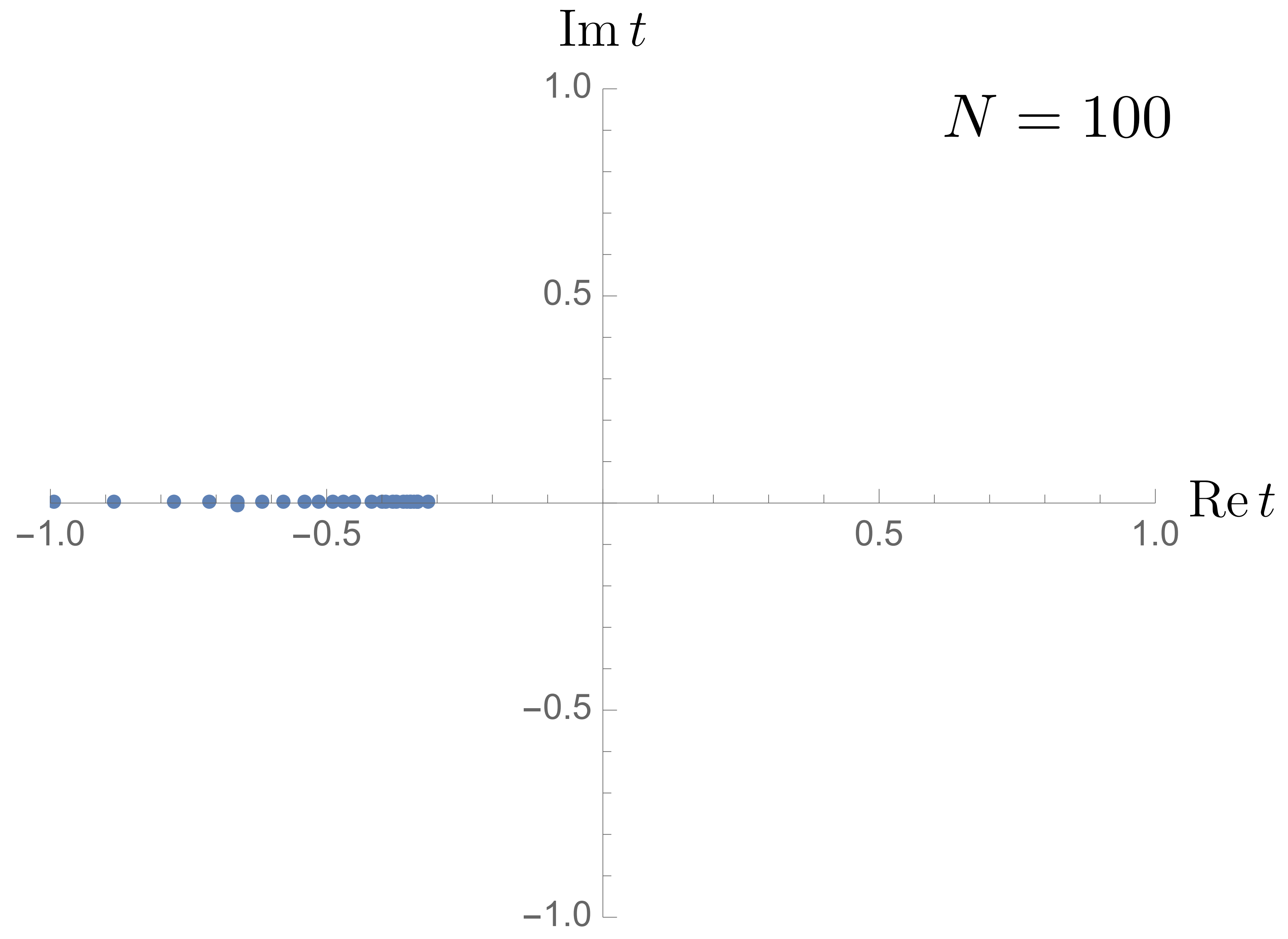
⚠ There are no singularities!

Padé-Borel transform: approximate a polynomial by a ratio of polynomials

$$\mathcal{P}[\mathcal{B}](t) = \frac{P_L(t)}{Q_M(t)} = \mathcal{B}(t) + \mathcal{O}(t^N)$$

$$L + M = N$$

The poles of $\mathcal{P}[\mathcal{B}](t)$ approximate the singularities of the full Borel transform



Anharmonic oscillator - solved

Perturbation theory done properly: $\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{x^2}{4}+\epsilon\frac{x^4}{4}\right)y(x;\epsilon)=E(\epsilon)y(x;\epsilon)$

STEP 1: $E(\epsilon)\sim\sum_{n=0}^{24}a_n\epsilon^n=\frac{1}{2}+\frac{3}{4}\epsilon-\frac{21}{8}\epsilon^2+\frac{333}{16}\epsilon^3-\frac{30885}{128}\epsilon^4+\frac{916731}{256}\epsilon^5-\frac{65518401}{1024}\epsilon^6+\ldots,\qquad \epsilon\to 0$

STEP 2: $a_n\sim(-1)^{n+1}\frac{\sqrt{6}}{\pi^{3/2}}3^n\Gamma\left(n+\frac{1}{2}\right),\qquad n\to\infty$

STEP 3: $\mathcal{B}(t)=\sum_{n=0}^{24}\frac{a_n}{n!}t^n$

STEP 4: $\mathcal{P}[\mathcal{B}](t)=\frac{P_{12}(t)}{Q_{13}(t)}=\sum_{n=0}^{24}\frac{a_n}{n!}t^n+\mathcal{O}(t^{25})$

STEP 5: $\mathcal{S}_E(\epsilon)=\frac{1}{\epsilon}\int_0^\infty \mathrm{d}t\, e^{-t/\epsilon}\mathcal{P}[\mathcal{B}](t)$

Inverse lifetime:

$$\mathrm{Im}\,\mathcal{S}_E(-\epsilon)\propto\Gamma\approx\exp\left(-\frac{1}{3\epsilon}\right)$$

Ground state energy:

$$\sum_{n=0}^{24}a_n=-1.45\times 10^{34}$$

Numerical integration : $E(1)=0.803770\ldots$

Borel summation : $\mathcal{S}_E(1)=0.803773\ldots$

Practice

We calculated the ground state energy for $\epsilon = 1$. Repeat this process for:

i) $\epsilon = 0.1$

ii) $\epsilon = 10$

A comprehensive Mathematica file on using Borel summation for the anharmonic oscillator can be found on my website: https://zachary-harris.grad.uconn.edu/research/

Using that file, you can numerically calculate the new ground state energies and generate as many perturbative coefficients as you want.

Do you need more or less than 25 terms in the perturbative expansion to achieve comparable precision?

Repeat this process for the first excited state, how does it compare?

Thank you!

More details on the precision of Borel resummation:

"Physical Resurgent Extrapolation" : arXiv:2003.07451

More physics applications:

"On the Higher Loop Euler-Heisenberg Trans-Series Structure" : arXiv:2101.10409

"Multi-instantons and Exact Results (I and II)" : arXiv:quant-ph/0501136 , arXiv:quant-ph/0501137

"Renormalons" : arXiv:hep-ph/9807443

Non-perturbative definition of path integrals:

"What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles" : arXiv:1511.05977

These slides are available on my website: https://zachary-harris.grad.uconn.edu/research/