1 Lagrangians

(1.1) Use Fermat's principle of least time to derive Snell's law.

The path taken by a beam of light always follows Fermat's principle of least time

$$T = \int_{t_A}^{t_B} \mathrm{d}t \quad \Rightarrow \quad \delta T = 0$$

If the path crosses two regions of space with different indices of refraction $n = c/v$, we can write the total time elapsed as

$$\begin{aligned} T &= \frac{1}{c} \int_A^B \mathrm{d}s \, \frac{\mathrm{d}t}{\mathrm{d}s} c \\ &= \frac{1}{c} \int_A^B \mathrm{d}s \, \frac{c}{v} \\ S &= \int_A^B \mathrm{d}s \, n \end{aligned}$$

where $S = cT$. Introducing a coordinate system for the plane in which the light beam moves, we can write

$$\begin{aligned} S &= \int_A^B \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} \, n(x,y) \\ &= \int_A^B \mathrm{d}s \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2} \, n(x,y) \\ S[y] &= \int_{x_A}^{x_B} \mathrm{d}x \, \sqrt{1 + \dot{y}^2} \, n(x,y) \\ S[y] &\equiv \int_{x_A}^{x_B} \mathrm{d}x \, L(y, \dot{y}, x) \end{aligned}$$

where $\dot{y} = \mathrm{d}y/\mathrm{d}x$ and $n(x,y)$ takes the constant values n_1 and n_2 on either side of the boundary which we can place along $x = 0$

$$n(x, y(x)) = \begin{cases} n_1 & x < 0 \\ n_2 & x > 0 \end{cases}$$

Computing $\delta S = 0$ (see problem 1.3), we find

$$\frac{\delta S[y]}{\delta y(x)} = \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial L}{\partial \dot{y}} = 0$$

which for our function takes the form

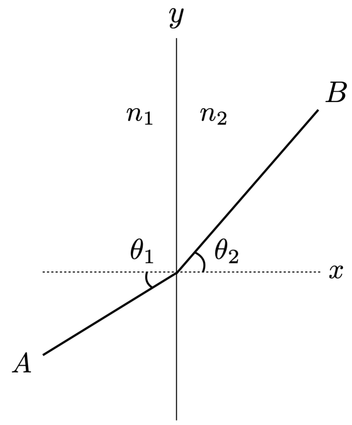
$$\sqrt{1 + \dot{y}^2} \, \frac{\partial n(x,y)}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} n(x,y) \right) = 0$$

Since $n(x,y)$ does not vary in the y-direction, the first term vanishes. In addition, the second term being a total derivative means we obtain

$$\begin{aligned} \frac{\dot{y}(x)}{\sqrt{1 + \dot{y}(x)^2}} n(x,y) &= \text{constant} \\ \frac{\dot{y}_1}{\sqrt{1 + \dot{y}_1^2}} n_1 &= \frac{\dot{y}_2}{\sqrt{1 + \dot{y}_2^2}} n_2 \\ \frac{\mathrm{d}y_1}{\sqrt{\mathrm{d}x_1^2 + \mathrm{d}y_1^2}} n_1 &= \frac{\mathrm{d}y_2}{\sqrt{\mathrm{d}x_2^2 + \mathrm{d}y_2^2}} n_2 \end{aligned}$$

Defining the angles of incidence and refraction, θ_1 and θ_2 respectively, relative to the line $y = 0$ (perpendicular to the boundary), we obtain Snell's law

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$



(1.2) Consider the functionals

$$H[f] = \int \mathrm{d}y\, G(x,y) f(y), \quad I[f] = \int_{-1}^{1} \mathrm{d}x\, f(x), \quad J[f] = \int \mathrm{d}y \left(\frac{\partial f}{\partial y}\right)^2$$

of the function f. Find the functional derivatives

$$\frac{\delta H[f]}{\delta f(z)}, \quad \frac{\delta^2 I[f^3]}{\delta f(x_0)\,\delta f(x_1)}, \quad \frac{\delta J[f]}{\delta f(x)}$$

Starting with $H[f]$, we find

$$\begin{aligned}
\frac{\delta H[f]}{\delta f(z)} &= \frac{\delta}{\delta f(z)} \int \mathrm{d}y\, G(x,y) f(y) \\
&= \int \mathrm{d}y\, G(x,y) \frac{\delta f(y)}{\delta f(z)} \\
&= \int \mathrm{d}y\, G(x,y) \delta(y-z) \\
\Aboxed{\frac{\delta H[f]}{\delta f(z)} &= G(x,z)}
\end{aligned}$$

(see problem 1.4 for going from line 2 to 3). Next, for $I[f]$ we find

$$\begin{aligned}
\frac{\delta I[f^3]}{\delta f(x_0)} &= \frac{\delta}{\delta f(x_0)} \int_{-1}^{1} \mathrm{d}x\, f(x)^3 \\
&= \int_{-1}^{1} \mathrm{d}x\, 3f(x)^2 \frac{\delta f(x)}{\delta f(x_0)} \\
&= \int_{-1}^{1} \mathrm{d}x\, 3f(x)^2 \delta(x - x_0) \\
\frac{\delta I[f^3]}{\delta f(x_0)} &= 3f(x_0)^2
\end{aligned}$$

and then

$$
\begin{aligned}
\frac{\delta^2 I[f^3]}{\delta f(x_0)\,\delta f(x_1)} &= \frac{\delta}{\delta f(x_1)} 3f(x_0)^2 \\
&= 6f(x_0)\frac{\delta f(x_0)}{\delta f(x_1)}
\end{aligned}
$$

$$
\boxed{\frac{\delta^2 I[f^3]}{\delta f(x_0)\,\delta f(x_1)} = 6f(x_0)\delta(x_0 - x_1)}
$$

Lastly, for $J[f]$ we find

$$
\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \frac{\delta}{\delta f(x)} \int \mathrm{d}y \left(\frac{\partial f}{\partial y}\right)^2 \\
&= \int \mathrm{d}y\, 2\frac{\partial f}{\partial y}\frac{\delta}{\delta f(x)}\left(\frac{\partial f}{\partial y}\right) \\
&= \int \mathrm{d}y\, 2\frac{\partial f}{\partial y}\frac{\partial}{\partial y}\left(\frac{\delta f(y)}{\delta f(x)}\right) \\
&= \int \mathrm{d}y\, 2\frac{\partial f}{\partial y}\frac{\partial}{\partial y}\delta(x-y) \\
&= -\int \mathrm{d}y\, 2\frac{\partial^2 f}{\partial y^2}\delta(x-y) + \cancelto{0}{2\frac{\partial f}{\partial y}\delta(x-y)\Big|_{\text{boundary}}}
\end{aligned}
$$

$$
\boxed{\frac{\delta J[f]}{\delta f(x)} = -2\frac{\partial^2 f}{\partial x^2}}
$$

(1.3) Consider the functional $G[f] = \int \mathrm{d}y\, g(y, f)$. Show that

$$
\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x, f)}{\partial f}
$$

Now consider the functional $H[f] = \int \mathrm{d}y\, g(y, f, f')$ and show that

$$
\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'}
$$

where $f' = \partial f/\partial y$. For the functional $J[f] = \int \mathrm{d}y\, g(y, f, f', f'')$ show that

$$
\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial f'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial g}{\partial f''}
$$

where $f'' = \partial^2 f/\partial y^2$.

To consider arbitrary functions inside a functional, we turn to the limit definition of the functional derivative

$$
\frac{\delta F[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{F[f(x') + \epsilon\delta(x' - x)] - F[f(x')]}{\epsilon}
$$

Applying this to the first functional $G[f]$, we find

$$\begin{aligned}
\frac{\delta G[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\,\Big(g(y,f+\epsilon\delta)-g(y,f)\Big)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\,\left(g(y,f)+\frac{\partial g(y,f)}{\partial f}\epsilon\delta(x-y)+\ldots-g(y,f)\right)\\
&= \int \mathrm{d}y\,\frac{\partial g(y,f)}{\partial f}\delta(x-y)\\
&\boxed{\frac{\delta G[f]}{\delta f(x)} = \frac{\partial g(x,f)}{\partial f}}
\end{aligned}$$

Applying this to $H[f]$, we find

$$\begin{aligned}
\frac{\delta H[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\,\Big(g(y,f+\epsilon\delta,f'+\epsilon\delta')-g(y,f,f')\Big)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\,\left(g(y,f,f')+\frac{\partial g(y,f,f')}{\partial f}\epsilon\delta(x-y)+\frac{\partial g(y,f,f')}{\partial f'}\epsilon\frac{\mathrm{d}}{\mathrm{d}y}\delta(x-y)+\ldots-g(y,f,f')\right)\\
&= \int \mathrm{d}y\,\left[\frac{\partial g}{\partial f}\delta(x-y)+\frac{\partial g}{\partial f'}\frac{\mathrm{d}}{\mathrm{d}y}\delta(x-y)\right]\\
&= \int \mathrm{d}y\,\left[\frac{\partial g}{\partial f}-\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial g}{\partial f'}\right)\right]\delta(x-y)+\cancelto{0}{\frac{\partial g}{\partial f'}\delta(x-y)\Big|_{\text{boundary}}}\\
&\boxed{\frac{\delta H[f]}{\delta f(x)} = \frac{\partial g}{\partial f}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial g}{\partial f'}\right)}
\end{aligned}$$

Lastly, applying this to $J[f]$, we find

$$\begin{aligned}
\frac{\delta J[f]}{\delta f(x)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\,\Big(g(y,f+\epsilon\delta,f'+\epsilon\delta',f''+\epsilon\delta'')-g(y,f,f',f'')\Big)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\,\left(g+\frac{\partial g}{\partial f}\epsilon\delta(x-y)+\frac{\partial g}{\partial f'}\epsilon\frac{\mathrm{d}}{\mathrm{d}y}\delta(x-y)+\frac{\partial g}{\partial f''}\epsilon\frac{\mathrm{d}^2}{\mathrm{d}y^2}\delta(x-y)+\ldots-g\right)\\
&= \int \mathrm{d}y\,\left[\frac{\partial g}{\partial f}\delta(x-y)+\frac{\partial g}{\partial f'}\frac{\mathrm{d}}{\mathrm{d}y}\delta(x-y)+\frac{\partial g}{\partial f''}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\delta(x-y)\right]\\
&= \int \mathrm{d}y\,\left[\frac{\partial g}{\partial f}-\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial g}{\partial f'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{\partial g}{\partial f''}\right)\right]\delta(x-y)+\cancelto{0}{\frac{\partial g}{\partial f'}\delta(x-y)\Big|_{\text{boundary}}}+\cancelto{0}{\frac{\partial g}{\partial f''}\frac{\mathrm{d}}{\mathrm{d}y}\delta(x-y)\Big|_{\text{boundary}}}\\
&\quad-\cancelto{0}{\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial g}{\partial f''}\right)\delta(x-y)\Big|_{\text{boundary}}}\\
&\boxed{\frac{\delta J[f]}{\delta f(x)} = \frac{\partial g}{\partial f}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial g}{\partial f'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial g}{\partial f''}\right)}
\end{aligned}$$

(1.4) Show that $\dfrac{\delta\phi(x)}{\delta\phi(y)}=\delta(x-y)$ and $\dfrac{\delta\dot\phi(t)}{\delta\phi(t_0)}=\dfrac{\mathrm{d}}{\mathrm{d}t}\delta(t-t_0)$.

Using the limit definition of the functional derivative and treating $\phi(x)$ as a trivial functional of ϕ, we directly obtain

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \lim_{\epsilon\to 0}\frac{\Big(\phi(x)+\epsilon\delta(x-y)\Big)-\phi(x)}{\epsilon}$$

$$\boxed{\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)}$$

Treating $\dot{\phi}(t)$ as a trivial functional of ϕ, we also obtain

$$\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi(t)+\epsilon\delta(t-t_0)\Big)-\dot{\phi}(t)\right]$$

$$\boxed{\frac{\delta\dot{\phi}(t)}{\delta\phi(t_0)} = \frac{\mathrm{d}}{\mathrm{d}t}\delta(t-t_0)}$$

(1.5) For a three-dimensional elastic medium, the potential and kinetic energy are

$$V = \frac{\mathcal{T}}{2}\int \mathrm{d}^3x\,(\boldsymbol{\nabla}\psi)^2, \qquad T = \frac{\rho}{2}\int \mathrm{d}^3x\left(\frac{\partial\psi}{\partial t}\right)^2$$

respectively. Use these results, and the functional derivative approach, to show that ψ obeys the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

where v is the velocity of the wave.

The action for this system can be expressed as the functional

$$S[\psi] = \int \mathrm{d}t\,L[\psi] = \int \mathrm{d}^4x\,\mathcal{L}(\dot{\psi},\boldsymbol{\nabla}\psi) = \int \mathrm{d}^4x\left(\frac{\rho}{2}\dot{\psi}^2 - \frac{\mathcal{T}}{2}(\boldsymbol{\nabla}\psi)^2\right)$$

where $\mathrm{d}^4x = \mathrm{d}t\,\mathrm{d}^3x$. For a multivariable function, the functional derivative is

$$\frac{\delta\psi(t,\mathbf{x})}{\delta\psi(t',\mathbf{x}')} = \delta^{(4)}(x-x') \equiv \delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

The equation of motion is defined by the condition that $\delta S = 0$, which can be expressed as

$$\begin{aligned}
\frac{\delta S[\psi]}{\delta\psi(t',\mathbf{x}')} &= \int \mathrm{d}^4x\left[\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right)\frac{\partial}{\partial t}\delta^{(4)}(x-x') + \frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)}\cdot\boldsymbol{\nabla}\delta^{(4)}(x-x')\right] \\
&= \int \mathrm{d}^4x\left[-\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) - \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)}\right)\right]\delta^{(4)}(x-x') + \cancelto{0}{\int \mathrm{d}^3x\,\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\delta^{(4)}(x-x')\Big|_{t\text{ boundary}}} \\
&\quad + \cancelto{0}{\int \mathrm{d}t\oint \mathrm{d}^2x\,\hat{\mathbf{n}}\cdot\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)}\delta^{(4)}(x-x')}
\end{aligned}$$

$$\frac{\delta S[\psi]}{\delta\psi(t,\mathbf{x})} = -\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) - \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)}\right) = 0$$

where in the last step we drop the primes. This yields the Euler-Lagrange equations of motion

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) + \boldsymbol{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)}\right) &= 0\\ \frac{\partial}{\partial t}\left(\rho\frac{\partial \psi}{\partial t}\right) - \boldsymbol{\nabla}\cdot(\mathcal{T}\boldsymbol{\nabla}\psi) &= 0\\ \nabla^2\psi - \frac{\rho}{\mathcal{T}}\frac{\partial^2\psi}{\partial t^2} &= 0\end{aligned}$$

If we define $v = \sqrt{\frac{\mathcal{T}}{\rho}}$ which has units $[v] = \frac{L}{T}$, we obtain the wave equation

$$\boxed{\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}}$$

(1.6) Show that if $Z_0[J]$ is given by

$$Z_0[J] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right)$$

where $\Delta(x) = \Delta(-x)$ then

$$\frac{\delta Z_0[J]}{\delta J(z_1)} = -\left[\int \mathrm{d}^4y\,\Delta(z_1-y)J(y)\right]Z_0[J]$$

The functional derivative of an exponential has the same structure as the ordinary derivative of an exponential

$$\frac{\delta}{\delta J(x)}e^{F[J]} = \frac{\delta F[J]}{\delta J(x)}e^{F[J]}$$

This can be seen by writing the exponential as a Taylor series

$$\begin{aligned}\frac{\delta}{\delta J(x)}e^{F[J]} &= \frac{\delta}{\delta J(x)}\sum_{n=0}^{\infty}\frac{1}{n!}F[J]^n\\ &= \sum_{n=0}^{\infty}\frac{1}{(n-1)!}F[J]^{n-1}\frac{\delta F[J]}{\delta J(x)}\\ \frac{\delta}{\delta J(x)}e^{F[J]} &= \frac{\delta F[J]}{\delta J(x)}e^{F[J]}\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\frac{\delta Z_0[J]}{\delta J(z_1)} &= \frac{\delta}{\delta J(z_1)}\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right)Z_0[J]\\ &= \left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\frac{\delta J(x)}{\delta J(z_1)}\Delta(x-y)J(y) - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)\frac{\delta J(y)}{\delta J(z_1)}\right)Z_0[J]\\ &= \left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\delta^{(4)}(x-z_1)\Delta(x-y)J(y) - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)\delta^{(4)}(y-z_1)\right)Z_0[J]\\ &= \left(-\frac{1}{2}\int \mathrm{d}^4y\,\Delta(z_1-y)J(y) - \frac{1}{2}\int \mathrm{d}^4x\,J(x)\Delta(x-z_1)\right)Z_0[J]\end{aligned}$$

$$\boxed{\frac{\delta Z_0[J]}{\delta J(z_1)} = \left(-\int \mathrm{d}^4y\,\Delta(y-z_1)J(y)\right)Z_0[J]}$$