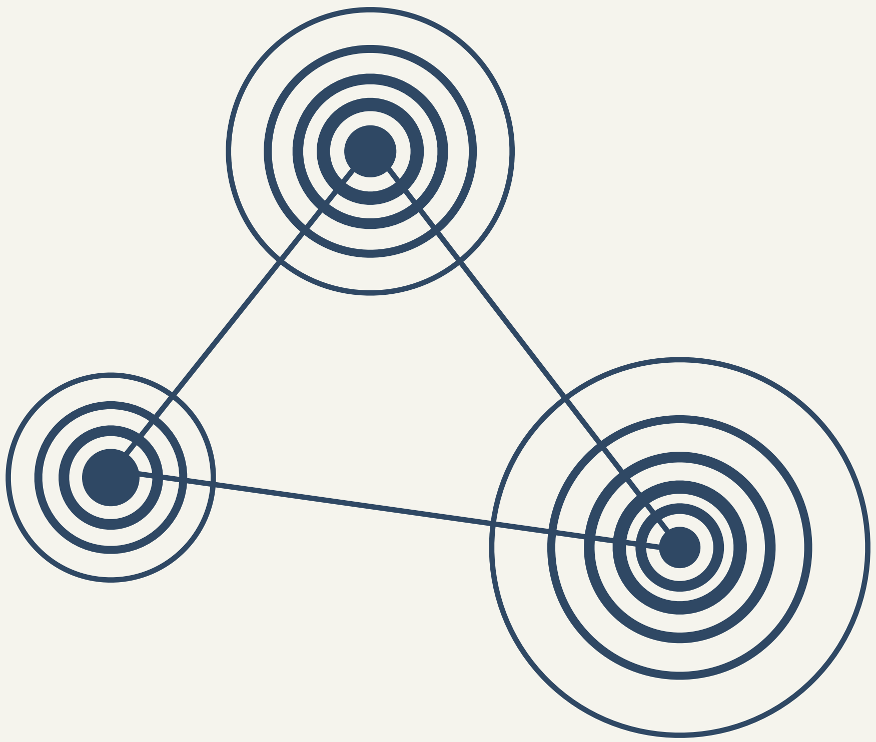

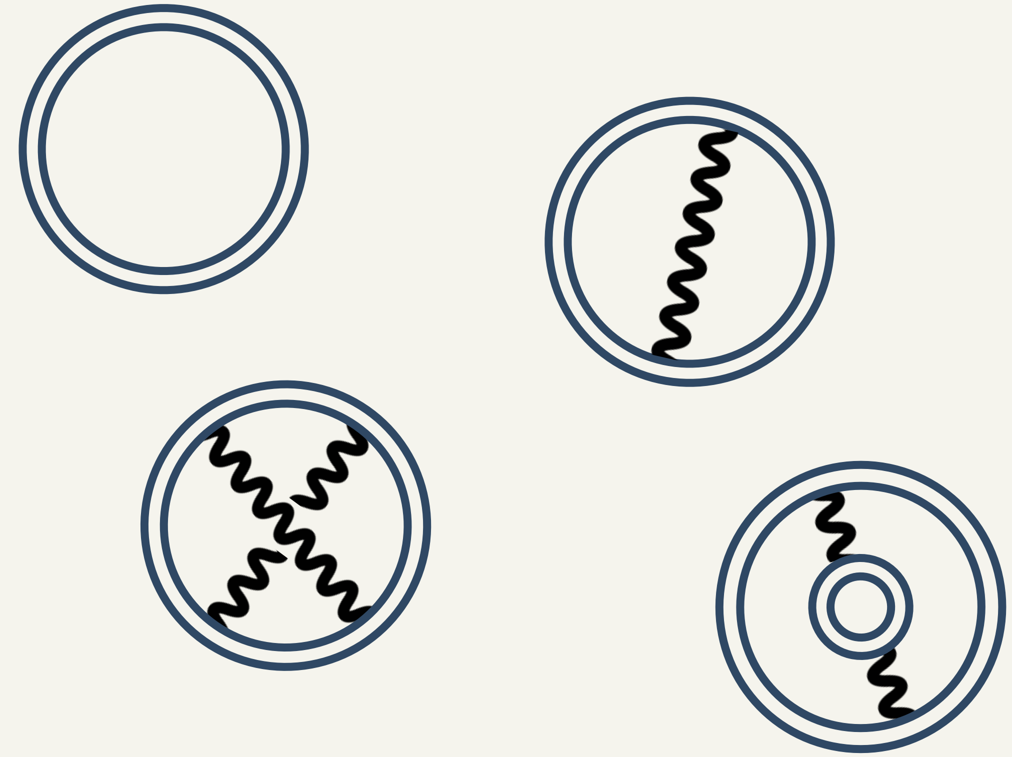

Resurgence and the Euler-Heisenberg Effective Action

Zachary Harris and Gerald Dunne

Physics Department, University of Connecticut

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Overview

- Resurgence and Borel summation
- Euler-Heisenberg effective action
- One-loop analysis
- Two-loop analysis
- Future work and conclusion

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Perturbation theory

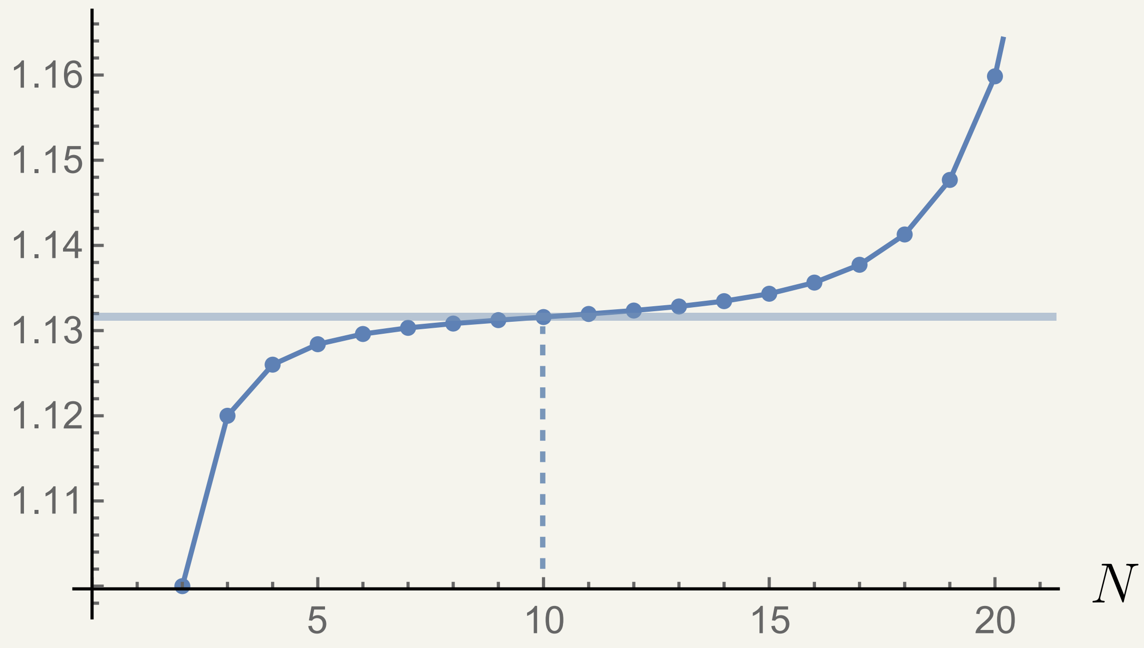
$$\mathcal{O}(g) = c_0 + c_1 g + c_2 g^2 + \dots$$

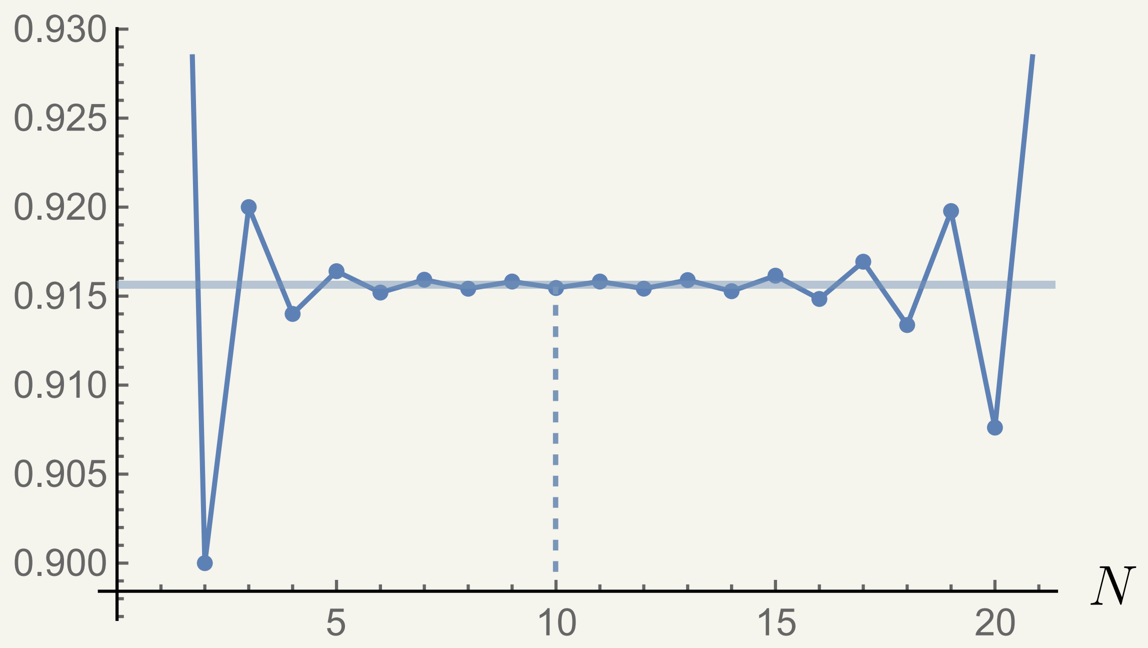
- Primary tool in QM and QFT: perturbation theory.
- ⚠ In most cases, $c_n \sim n!$ $\Rightarrow$ the sum diverges for all g
- E.g. ground state energy $E_0(g)$:
  - Zeeman effect: $c_n \sim (-1)^n (2n)!$ Stable
  - Stark effect: $c_n \sim (2n)!$ Unstable

Perturbation theory

- Why does perturbation theory work?
  - Asymptotic series

$$\sum_{n=0}^{N} n!x^n \qquad (x = 0.1) \qquad \sum_{n=0}^{N} (-1)^n n!x^n$$

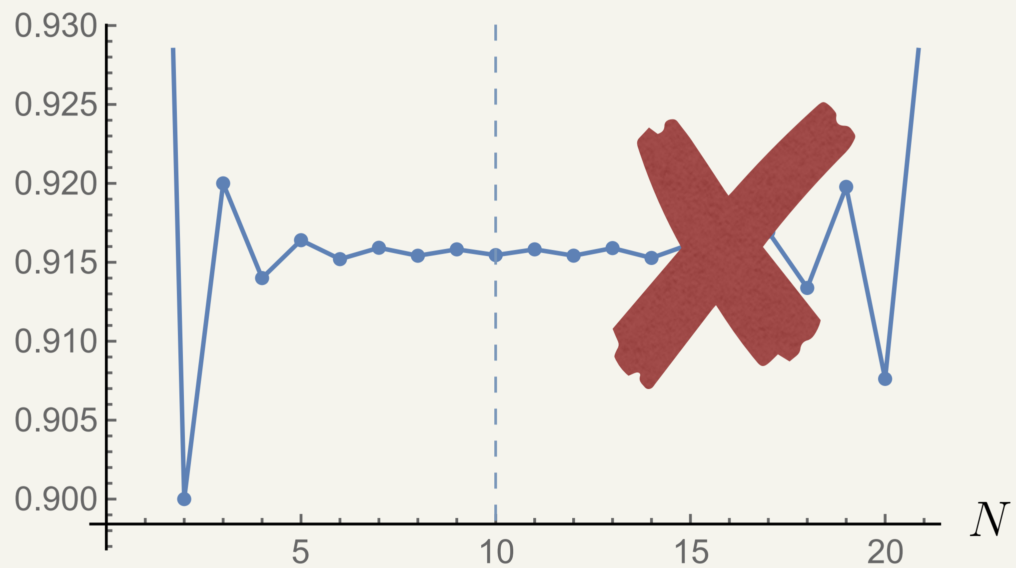




- Optimal truncation: truncate series at order $N \sim 1/x$

Perturbation theory

- What's wrong with just truncating?



- If $f(x) = \sum_{n=0}^{N-1} c_n x^n + R_N(x)$, optimally truncating yields an error

$$|R_N(x)|_{N \approx 1/x} \approx \frac{e^{-1/x}}{\sqrt{x}}$$

- This is precisely the scale of non-perturbative phenomena.

- We need a more subtle way to extract information.

Borel summation

- For a given asymptotic series, construct the Borel transform function

$$f(g) \stackrel{?}{=} \sum_{n=0}^{\infty} c_n g^n, \quad c_n \sim n! \qquad \Rightarrow \qquad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

- By dividing out the leading growth, the Borel transform function will have a non-zero radius of convergence.

- The Borel sum of $f(g)$ is given by the Laplace integral (if it exists)

$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathrm{d}t \, e^{-t/g} \mathcal{B}[f](t) \sim \sum_{n=0}^{\infty} c_n g^n, \quad g \to 0$$

- Big picture: singularities of the Borel transform encode non-perturbative information which isn't directly accessible from the perturbative expansion.

Borel summation

- Example: pure factorial growth

$$f(g) = \sum_{n=0}^{\infty} (-1)^n n! g^n \quad \Rightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} (-1)^n t^n = \frac{1}{1+t}$$

- Borel transform has one singularity off the integration contour, so the Borel sum is well-defined and unambiguous in the right-half complex g plane

$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathrm{d}t \, \frac{1}{1+t} e^{-t/g} = \frac{e^{1/g}}{g} \Gamma\left(0, \tfrac{1}{g}\right), \quad \mathrm{Re}\, g > 0$$

- The global analytic properties of the incomplete gamma function are well-known, including it's asymptotic expansion as $g \to 0$

$$\mathcal{S}f(g) \sim \frac{e^{1/g}}{g} e^{-1/g} \left(g - g^2 + 2g^3 - 6g^4 + 24g^5 - \ldots\right) = \sum_{n=0}^{\infty} (-1)^n n! g^n$$

Borel summation

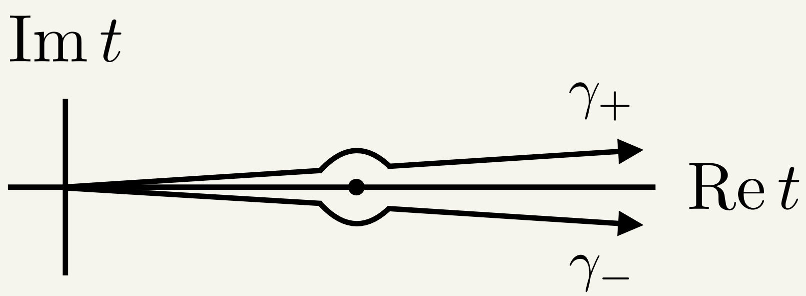
- What about $\mathrm{Re}\, g < 0$? The coefficients are now non-alternating

$$f(-g) = \sum_{n=0}^{\infty} n! g^n \quad \Rightarrow \quad \mathcal{B}[f](t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$$

- ⚠ Borel transform has a pole on the integration contour

$$\mathcal{S}f(-g) = -\frac{1}{g}\int_0^{\infty} \mathrm{d}t\, \frac{1}{1-t} e^{-t/g} \stackrel{?}{=} -\frac{e^{-1/g}}{g}\Gamma\left(0, -\tfrac{1}{g}\right)$$

- Need a prescription to deform the contour



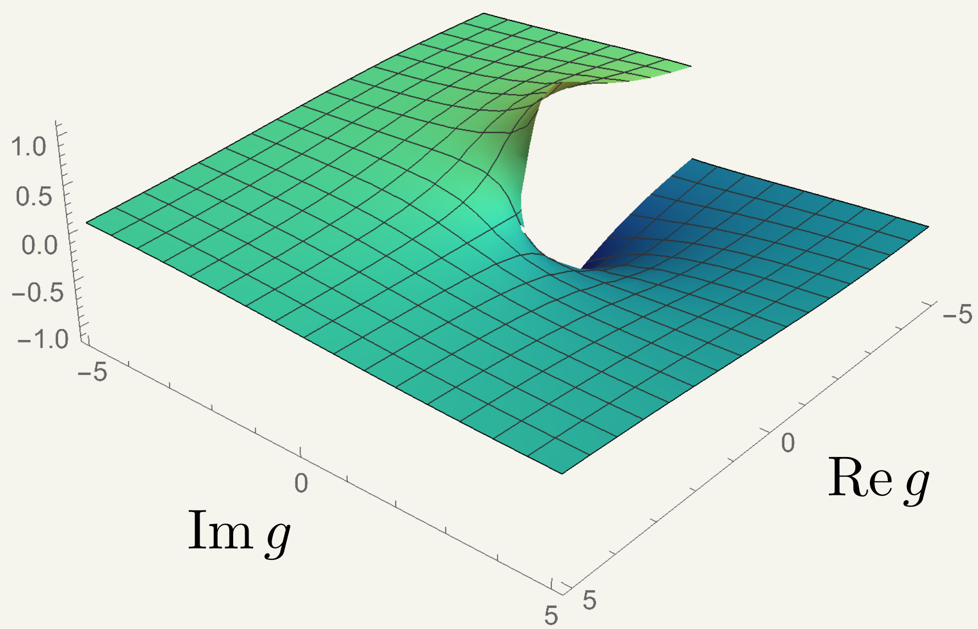
$$\mathrm{Re}\,[\mathcal{S}f(-g)] = \frac{1}{2}\left(\int_{\gamma_+} + \int_{\gamma_-}\right) = -\frac{1}{g}\int_0^{\infty} \mathrm{d}t\, \frac{1}{1-t} e^{-t/g}$$

$$\mathrm{Im}\,[\mathcal{S}f(-g)] = \pm\frac{1}{2}\left(\int_{\gamma_+} - \int_{\gamma_-}\right) = \pm\frac{\pi}{g} e^{-1/g}$$

Borel summation

- Does this agree with the symbolic result $-\frac{e^{-1/g}}{g}\Gamma\left(0,-\frac{1}{g}\right)$? Yes!

$$\mathrm{Im}\left[\frac{e^{1/g}}{g}\Gamma\left(0,\tfrac{1}{g}\right)\right]$$



Connection formula:

$$\frac{1}{2}\left[\Gamma\left(0,\tfrac{1}{g}e^{i\pi}\right)-\Gamma\left(0,\tfrac{1}{g}e^{-i\pi}\right)\right]=-i\pi$$

$$\Downarrow$$

$$\mathcal{S}f(-g)=\mathrm{Re}\left[\mathcal{S}f(-g)\right]+\frac{i\pi}{g}e^{-1/g}$$

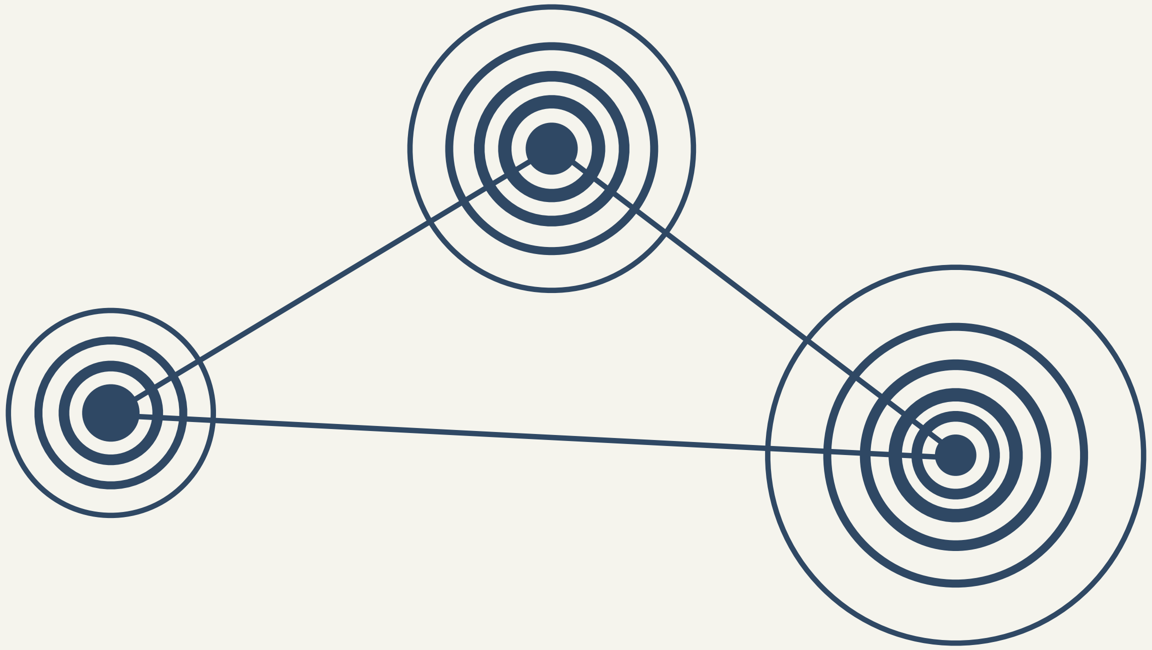
- The singularities in the Borel plane encode:
  - the global analytic structure of the Borel sum.
  - imaginary, non-perturbative contributions to the Borel sum which were inaccessible from the original asymptotic expansion.

Resurgence

- Going beyond just perturbation theory, observables should have the generic structure of "trans-series"

$$\mathcal{O}(g) = \sum_k \sum_\ell \sum_m \underbrace{c_{k\ell m} g^m}_{\substack{\text{perturbative}\\ \text{fluctuations}}} \underbrace{\left(\exp\left[-\frac{1}{g}\right]\right)^\ell}_{\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g}\right]\right)^k}_{\text{quasi-zero modes}}$$

- Resurgence: the perturbative fluctuations and the fluctuations about the non-trivial "saddles" are related.



Resurgence

- What if we only have a finite amount of perturbative data?

$$\mathcal{O}_N(g) = \sum_{n=0}^{N-1} c_n g^n, \quad c_n \sim n! \qquad \Rightarrow \qquad \mathcal{B}_N[\mathcal{O}](t) = \sum_{n=0}^{N-1} \frac{c_n}{n!} t^n$$

- The Borel transform function is a polynomial (i.e. no singularities).

- Solution: model poles with finite precision:

  - Padé approximant: $\mathcal{P}\mathcal{B}_N[\mathcal{O}](t) = \dfrac{P_L(t)}{Q_M(t)} = \mathcal{B}_N[\mathcal{O}](t) + O(t^{N+1}), \quad L + M = N$

  - Conformal map + Padé: separates branch points.

  - Uniformization map + Padé: separates branch points with higher precision.

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Euler-Heisenberg effective action

- QED (natural units, $\hbar = c = \varepsilon_0 = 1$)

$$Z[J] = \int \mathcal{D}[A]\mathcal{D}[\psi]\mathcal{D}[\bar{\psi}]\ \exp\left(i\int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D}[A] - m)\psi + J_\mu A^\mu\right)\right)$$

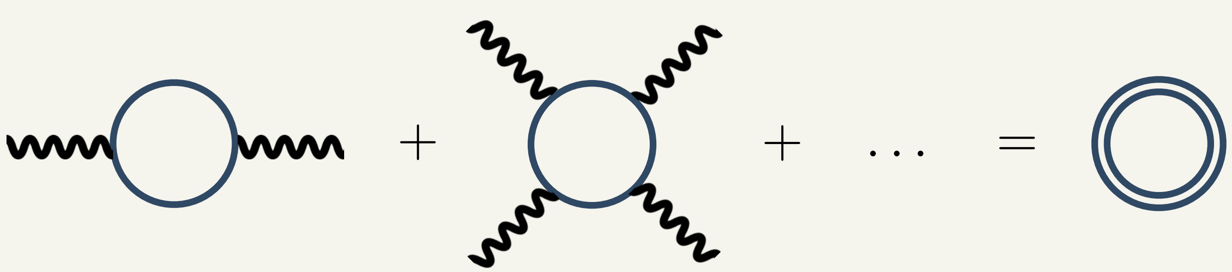
- "Integrate out" fermions ($E_\gamma \ll m_e$) and expand in fluctuations about a classical electromagnetic field configuration ($A = \bar{A} + q$, $J^\mu = \partial_\nu F^{\mu\nu}[\bar{A}]$)

$$Z[\bar{A}] = e^{-iS_{\text{Max.}}[\bar{A}]}\int \mathcal{D}[q]\ \exp\left(i\int \mathrm{d}^4x\left(-\frac{1}{4}Q_{\mu\nu}Q^{\mu\nu}\right) + \ln\det\left(i\not{D}[\bar{A}+q] - m\right)\right)$$

- Consider the quantity $W = -i\ln Z$. Expanding the functional determinant in powers of q, we obtain

$$W[\bar{A}] = -S_{\text{Maxwell}}[\bar{A}] + \Gamma_{\text{EH}}[\bar{A}], \qquad \Gamma_{\text{EH}}[\bar{A}] = \sum_{\ell=1}^{\infty}\Gamma_{\text{EH}}^{(\ell)}[\bar{A}]$$

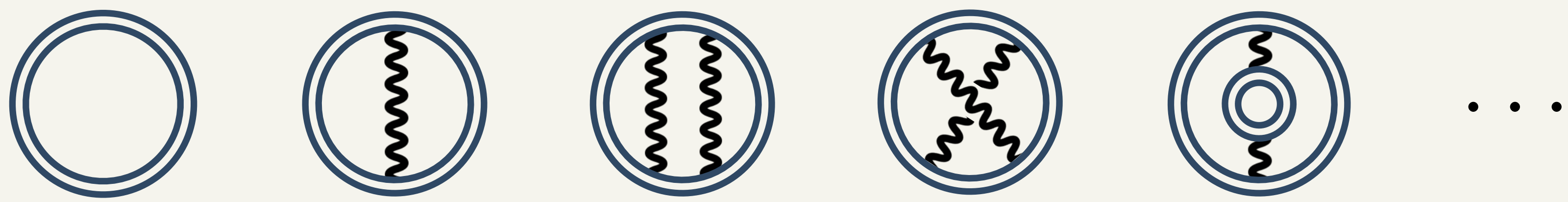
$$\Gamma_{\text{EH}}^{(1)}[\bar{A}] = -i\ln\det\left(i\not{D}[\bar{A}] - m\right) =$$



Euler-Heisenberg effective action

- The effective action can be written in terms of an effective Lagrangian

$$\Gamma_{\rm EH}[F] = \int \mathrm{d}^4x\, \mathcal{L}_{\rm EH}\left(\alpha, \frac{eF}{m^2}\right), \qquad \mathcal{L}_{\rm EH}\left(\alpha, \frac{eF}{m^2}\right) \sim \sum_{\ell=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^{\ell} \mathcal{L}^{(\ell)}\left(\frac{eF}{m^2}\right)$$



- F is the strength of the constant background field, magnetic or electric.

- At each loop order, we can make a weak field expansion of the Lagrangian in powers of F^2 $\Rightarrow$ asymptotic series

$$\mathcal{L}^{(\ell)}\left(\frac{eF}{m^2}\right) \sim \frac{\pi^{2(\ell-2)} F^2}{(\ell-1)!} \left(\frac{eF}{m^2}\right)^2 \sum_{n=0}^{\infty} a_n^{(\ell)} \left(\frac{eF}{m^2}\right)^{2n}, \quad eF \ll m^2$$

$$a_n^{(\ell)} \;\rightarrow\; \text{perturbative data}$$

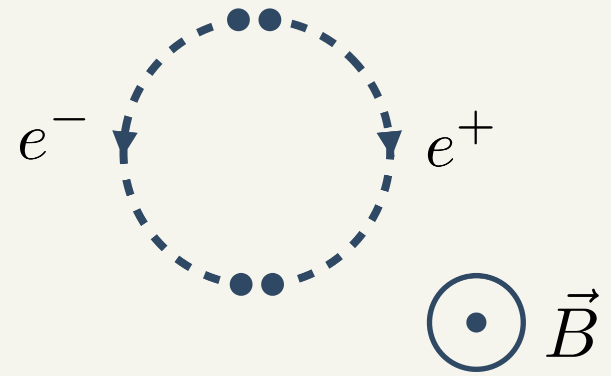
Euler-Heisenberg effective action

- Lorentz invariant

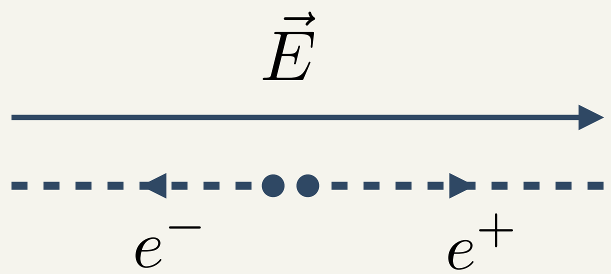
$$\mathcal{F} = -\frac{1}{2}\left(E^2 - B^2\right)$$

  - The change from magnetic to electric field should be possible through analytic continuation $B \to iE$ (recall $g \to -g$).

- Effects on particles



(Recall Zeeman effect)



(Recall Stark effect)

  - Schwinger mechanism: pair-production from polarizing the vacuum.

  - Non-perturbative effect $\sim e^{-1/E}$ (need large E for a rate near unity).

Euler-Heisenberg effective action

- Goals:
  - With a finite number of terms in the weak magnetic field perturbative expansion, use Borel summation to extrapolate to strong magnetic field.
    - Physical application of Borel summation and resurgence.
  - Analytically continue $B \to iE$ and study how the singularities in the Borel plane encode the non-perturbative Schwinger mechanism.
    - At two-loop, a closed-form expression for the pair-production rate is only partially known.

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One-loop analysis

- One-loop Euler-Heisenberg effective Lagrangian for a constant background magnetic field, B:

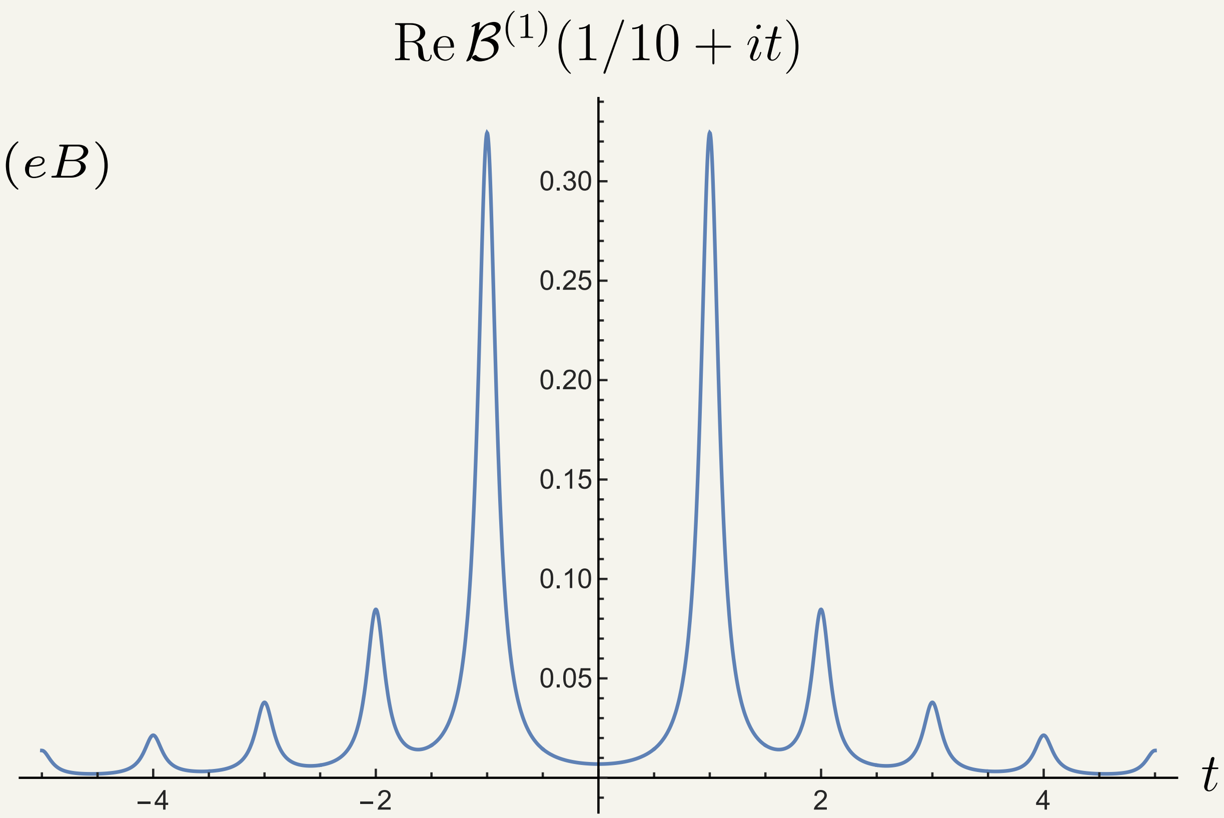
$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) = -\frac{B^2}{2}\int_0^\infty \frac{\mathrm{d}t}{t^2}\left(\coth t - \frac{1}{t} - \frac{t}{3}\right)e^{-m^2 t/(eB)}$$

- Already in the form of a Borel sum:

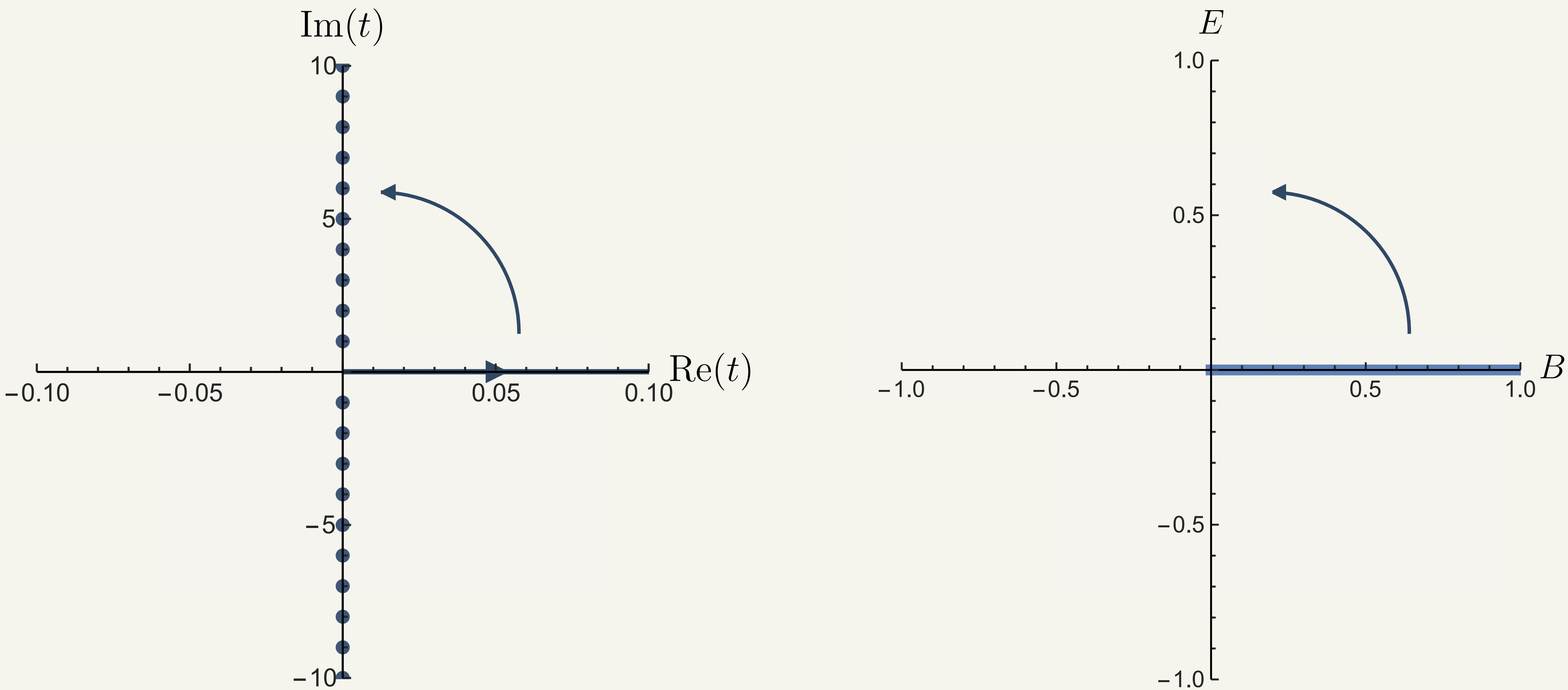
$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) = \frac{\pi B^2}{2}\int_0^\infty \mathrm{d}t\,\mathcal{B}^{(1)}(t)e^{-m^2\pi t/(eB)}$$

$$\mathcal{B}^{(1)}(t) = -\frac{1}{\pi^2 t^2}\left[\coth(\pi t) - \frac{1}{\pi t} - \frac{\pi t}{3}\right]$$

$$\mathcal{B}^{(1)}(t) = \frac{2}{\pi^3}\sum_{k=1}^{\infty}\frac{t}{k^2(t^2+k^2)}$$



One-loop analysis



- Imaginary part of one-loop Lagrangian with constant electric field:

$$\mathrm{Im}\,\mathcal{L}^{(1)}\left(\frac{eE}{m^2}\right) = \frac{E^2}{2\pi}\left(e^{-\pi m^2/(eE)} + \frac{1}{2^2}e^{-2\pi m^2/(eE)} + \frac{1}{3^2}e^{-3\pi m^2/(eE)} + \dots\right)$$

One-loop analysis

- Finite-order Borel analysis: weak field expansion $\left(eB \ll m^2\right)$

$$\mathcal{L}_N^{(1)}\left(\frac{eB}{m^2}\right) \sim \frac{B^2}{\pi^2}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{N} a_n^{(1)} \left(\frac{eB}{m^2}\right)^{2n}$$

$$a_n^{(1)} = (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}} \zeta(2n+4)$$

$$a_n^{(1)} = (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}} \left(1 + \frac{1}{2^2}\frac{1}{2^{2n+2}} + \frac{1}{3^2}\frac{1}{3^{2n+2}} + \dots\right)$$

- Borel transform:

$$\mathcal{B}_N^{(1)}(t) = \frac{2}{\pi^2} \sum_{n=0}^{N-1} \frac{a_n^{(1)}}{(2n+1)!} (\pi t)^{2n+1}$$

- Padé approximant: ⚠ What should L and M be?

$$\mathcal{PB}_N^{(1)}(t) = \frac{P_L^{(1)}(t)}{Q_M^{(1)}(t)} = \mathcal{B}_N^{(1)}(t) + O(t^N)$$

One-loop analysis

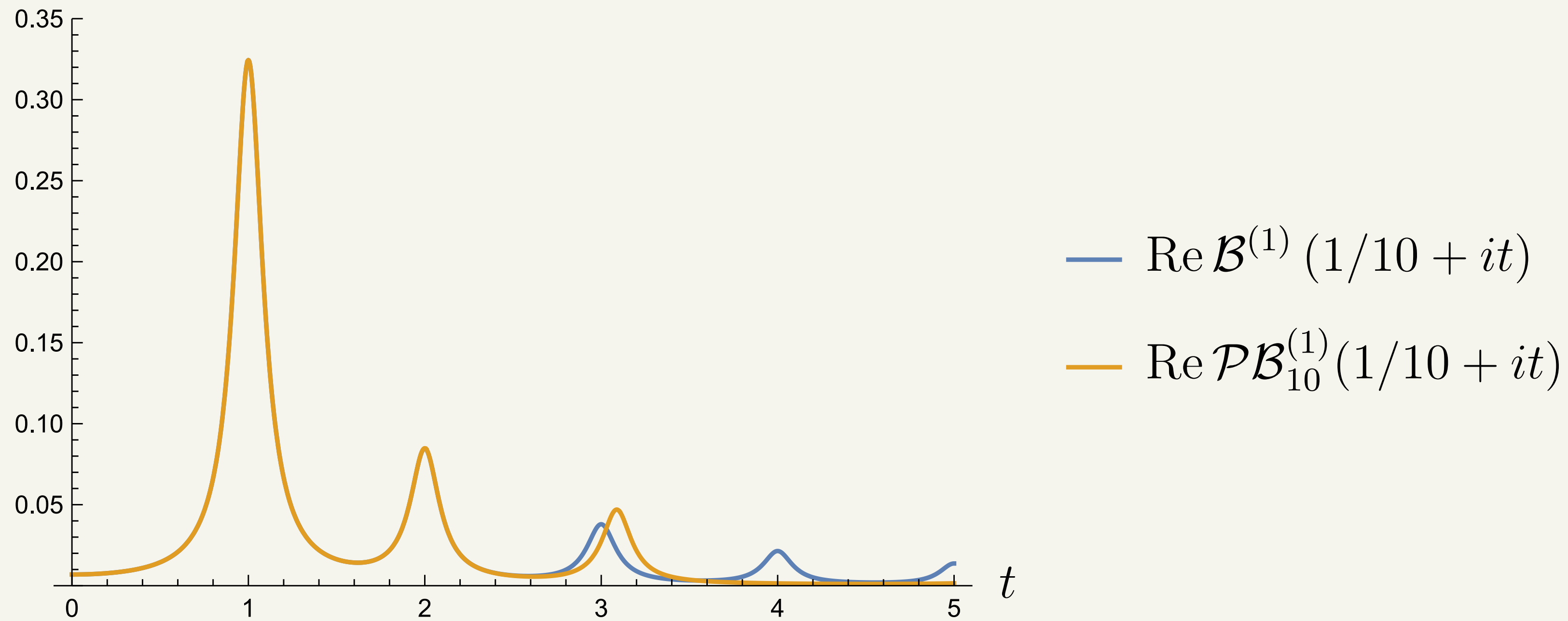
- We can enhance the Borel summation by encoding the strong field behavior ($eB \gg m^2$) into the Padé approximant:

$$\mathcal{L}^{(1)}\left(\frac{eB}{m^2}\right) \sim \frac{1}{3} \cdot \frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma + \frac{6}{\pi^2}\zeta'(2)\right)$$

$$\beta_{\text{QED}} = 2\alpha \sum_{n=1}^{\infty} \beta_n \left(\frac{\alpha}{\pi}\right)^n = 2\alpha\left[\frac{1}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2 + \dots\right]$$

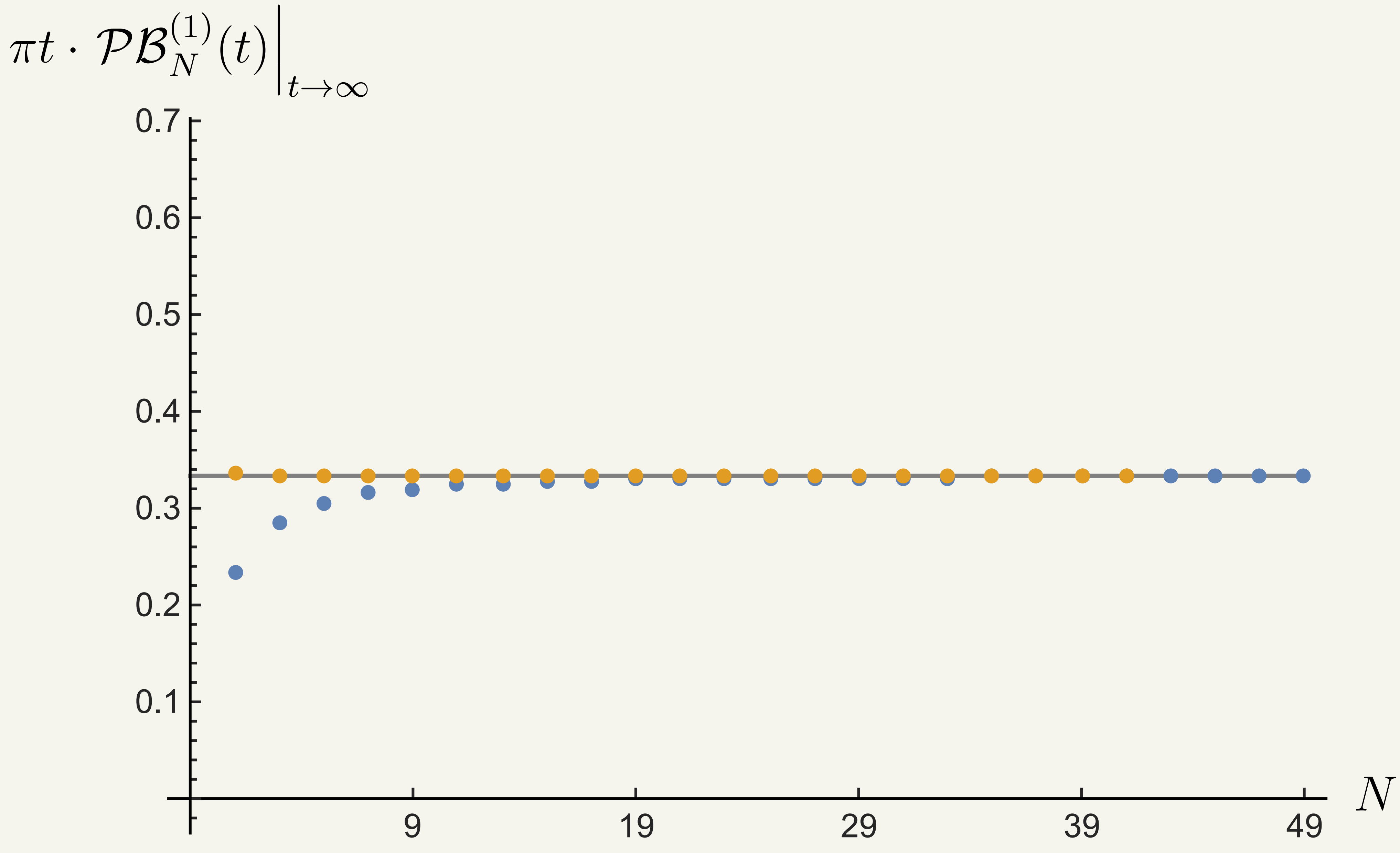
Euler-Mascheroni constant

- Condition: for large t, the Padé should go like $\mathcal{PB}_N^{(1)}(t \gg 1) \sim \dfrac{1}{\pi t}$



One-loop analysis

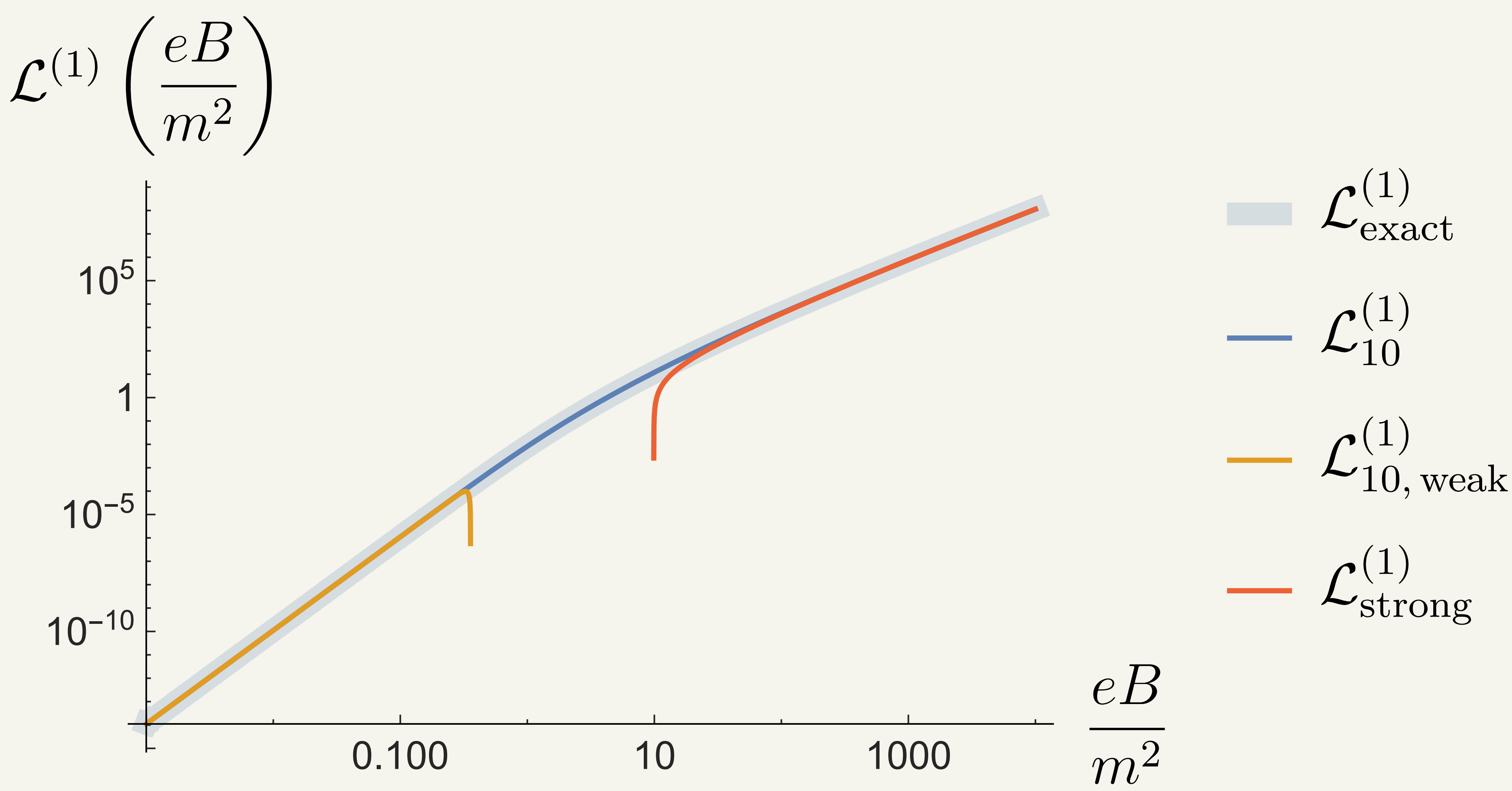
- We do not force that the coefficient be β_1 , this emerges from the data



One-loop analysis

- Borel summation of the one-loop magnetic field Lagrangian

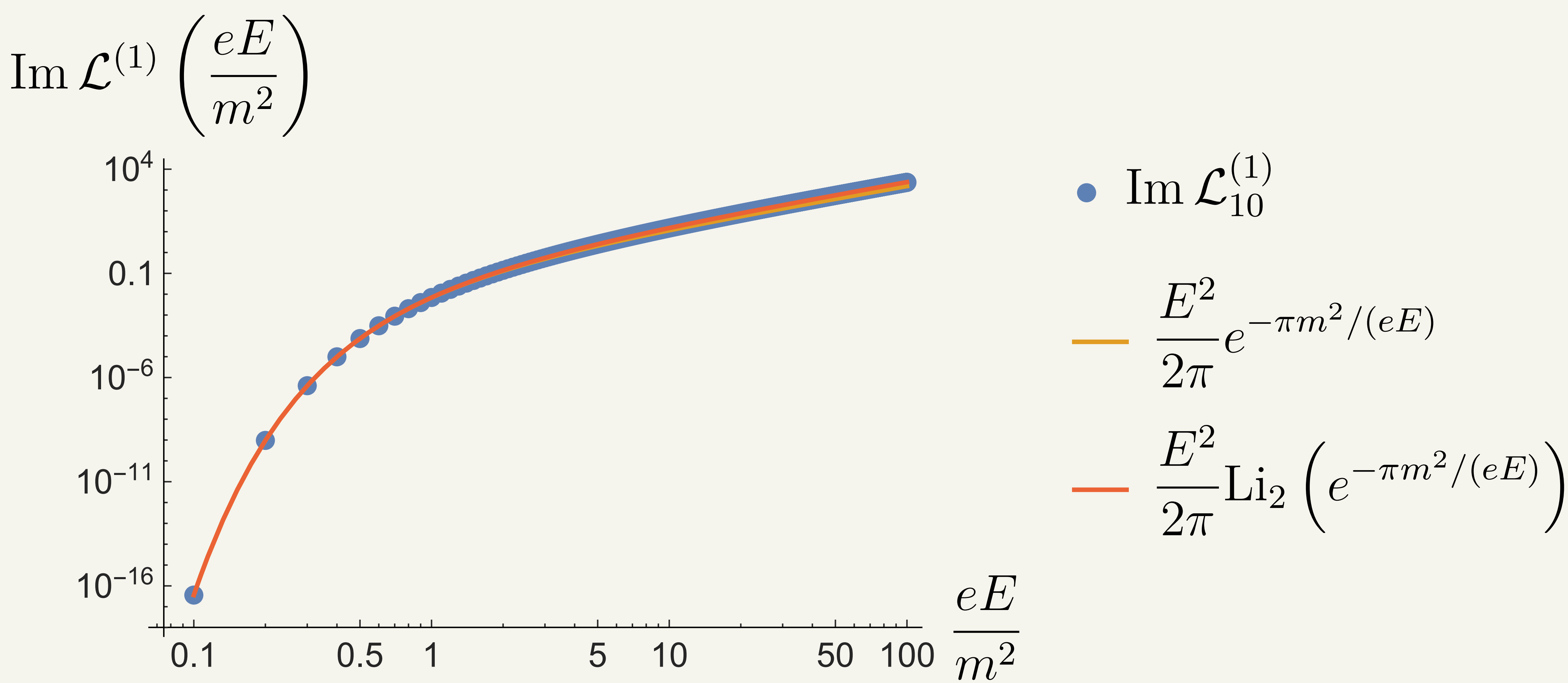
$$\mathcal{L}_N^{(1)}\left(\frac{eB}{m^2}\right) = \frac{\pi B^2}{2}\int_0^\infty \mathrm{d}t\,\mathcal{P}\mathcal{B}_N^{(1)}(t)e^{-\pi m^2 t/(eB)}$$



One-loop analysis

- Analytic continuation to an electric field configuration

$$\mathcal{L}_N^{(1)}\left(\frac{eE}{m^2}\right) = -\frac{\pi E^2}{2}\int_0^{e^{\pm i\varepsilon}\infty} i\,\mathrm{d}t\,\mathcal{P}\mathcal{B}_N^{(1)}(it)e^{-\pi m^2 t/(eE)}$$



One-loop analysis

- Summary:
  - Using only the first 10 terms in the weak magnetic field expansion, we were able to:
    - Recover the first beta function coefficient by encoding the leading strong field behavior.
    - Extrapolate from weak magnetic field to strong magnetic field over eight orders of magnitude.
    - Analytically continue to a constant electric field and obtain the exact pair production rate at one-loop (an infinite sum of non-perturbative contributions).
- Can the same be done at two-loop?

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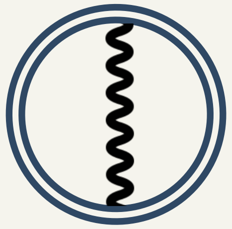
Two-loop analysis

- Two-loop Euler-Heisenberg effective action

$$\Gamma^{(2)}_{\mathrm{EH}}[F] =$$



- New features at two loop: mass renormalization and virtual interaction within the fermion loop.

- Effective Lagrangian: $2(\ell - 1) \to$ 2-parameter integral

$$\mathcal{L}^{(2)}\left(\frac{eB}{m^2}\right) = \frac{B^2}{4}\int_0^\infty \frac{\mathrm{d}t}{t^3} e^{-t\,m^2/(eB)}(J_1 + J_2 + J_3)$$

$$J_1 = \frac{2tm^2}{eB}\int_0^1 \frac{\mathrm{d}s}{s(1-s)}\left[\frac{\cosh(t\,s)\cosh(t(1-s))}{a-b}\ln\frac{a}{b} - t\coth t + \frac{5t^2}{6}s(1-s)\right]$$

$$J_2 = -\int_0^1 \frac{\mathrm{d}s}{s(1-s)}\left[\frac{c}{(a-b)^2}\ln\frac{a}{b} - \frac{1 - b\cosh(t(1-2s))}{b(a-b)} + \frac{b\cosh t + 1}{2b^2} - \frac{5t^2}{6}s(1-s)\right]$$

$$J_3 = \left(1 + 3\frac{tm^2}{eB}\left(\ln\left(\frac{tm^2}{eB}\right) + \gamma - \frac{5}{6}\right)\right)\left(t\coth t - 1 - \frac{t^2}{3}\right)$$

Two-loop analysis

- Known/conjectured results:

  - Strong field behavior ($eB \gg m^2$)

$$\mathcal{L}^{(2)}\left(\frac{eB}{m^2}\right) \sim \underset{\displaystyle\downarrow\atop\displaystyle\beta_2}{\frac{1}{4}} \cdot \frac{B^2}{2}\left(\ln\left(\frac{eB}{\pi m^2}\right) - \gamma - \frac{5}{6} + 4\zeta(3)\right)$$

  - Imaginary part of electric field Lagrangian

$$\operatorname{Im}\mathcal{L}^{(2)}\left(\frac{eE}{m^2}\right) \sim \frac{\pi E^2}{2}\left\{e^{-\pi m^2/(eE)}\left(1+\dots\right) + \sqrt{\frac{m^2}{eE}}\sum_{k=2}^{\infty} e^{-k\pi m^2/(eE)}\left(-c_k + \sqrt{\frac{eE}{m^2}} + \dots\right)\right\}$$

$$c_k = \frac{1}{2\sqrt{k}}\sum_{l=1}^{k-1}\frac{1}{\sqrt{l(k-l)}}, \quad k \geq 2$$

Two-loop analysis

- Finite-order Borel analysis: 50 terms in weak field expansion

$$\mathcal{L}_N^{(2)}\left(\frac{eB}{m^2}\right) \sim B^2\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{N-1} a_n^{(2)}\left(\frac{eB}{m^2}\right)^{2n}$$

$$a_n^{(2)} \sim (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}} + \text{corrections}$$

- Borel transform:

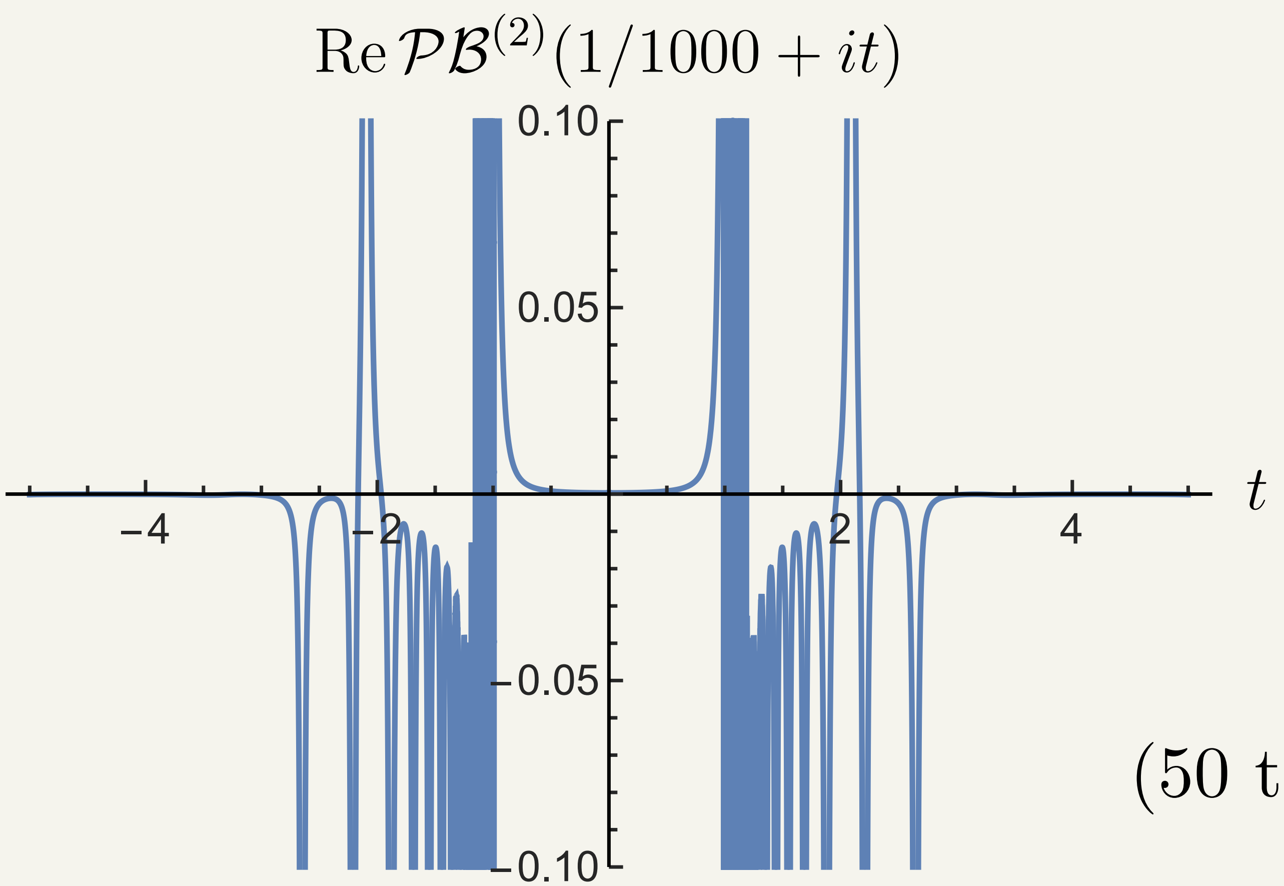
$$\mathcal{B}_N^{(2)}(t) = 2\sum_{n=0}^{N-1} \frac{a_n^{(2)}}{(2n+1)!}(\pi t)^{2n+1}$$

- Padé approximant:

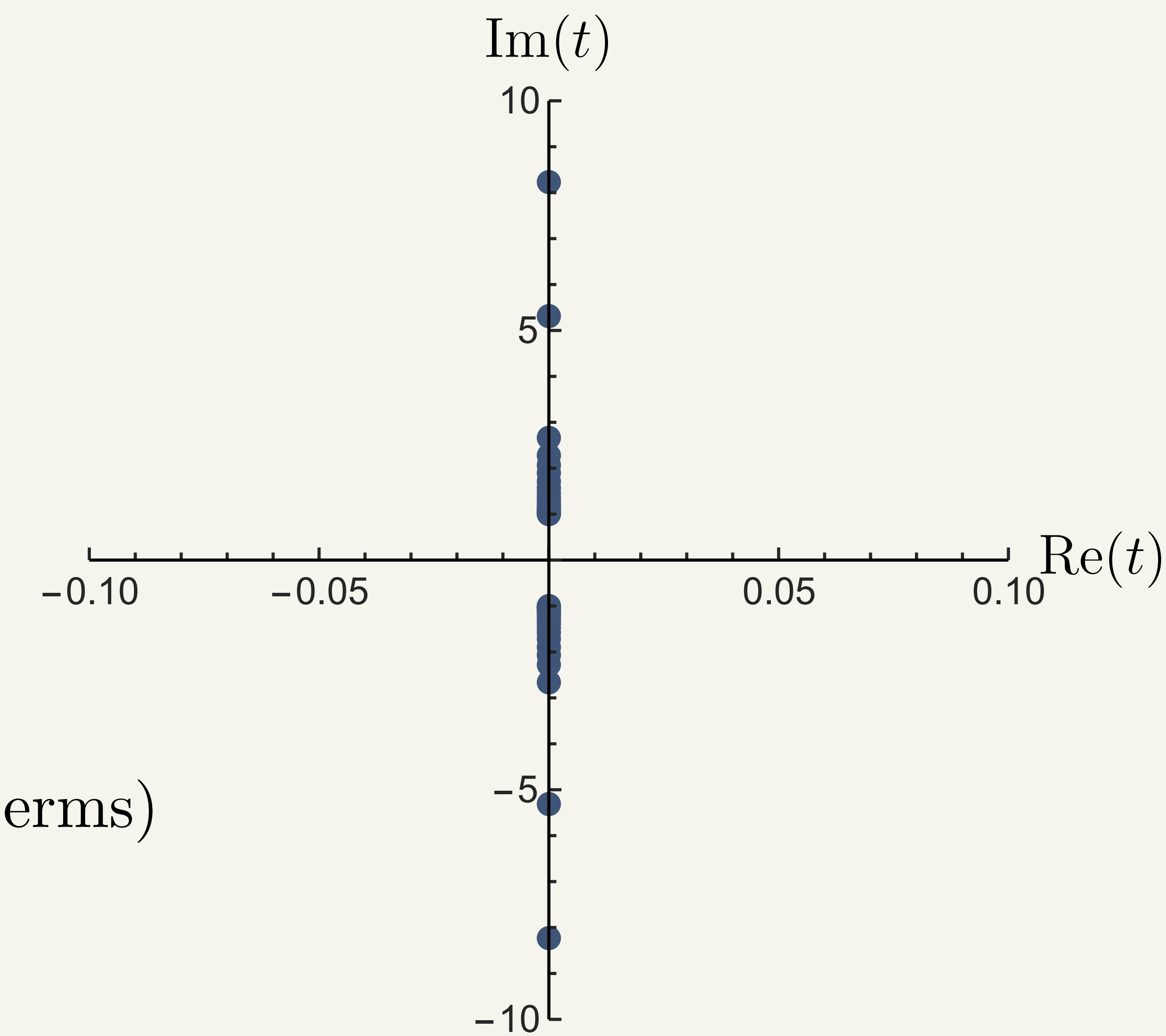
$$\mathcal{P}\mathcal{B}_N^{(2)}(t) = \frac{P_N^{(2)}(t)}{Q_{N+1}^{(2)}(t)}$$

Two-loop analysis

- Instead of integer spaced poles, the Borel plane has branch cuts



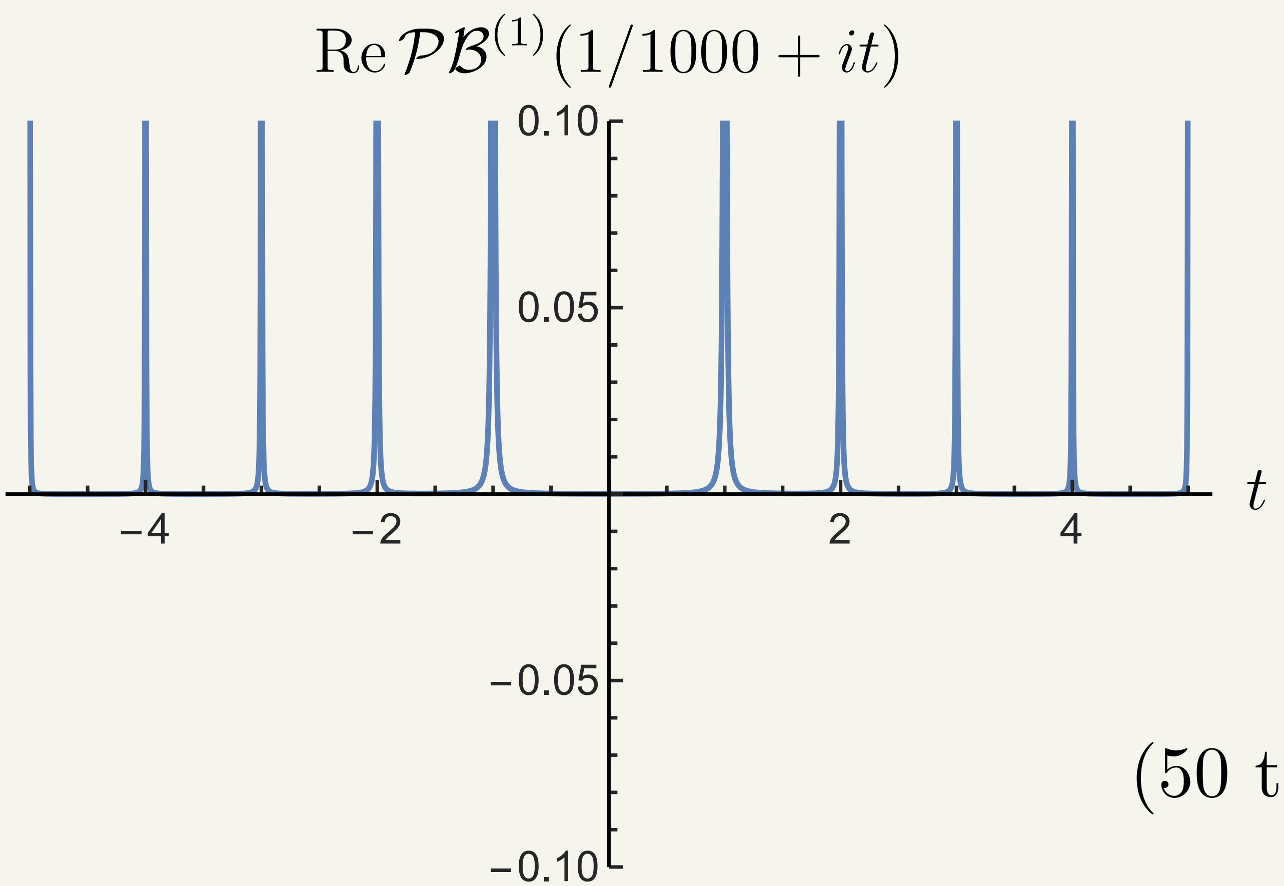
(50 terms)



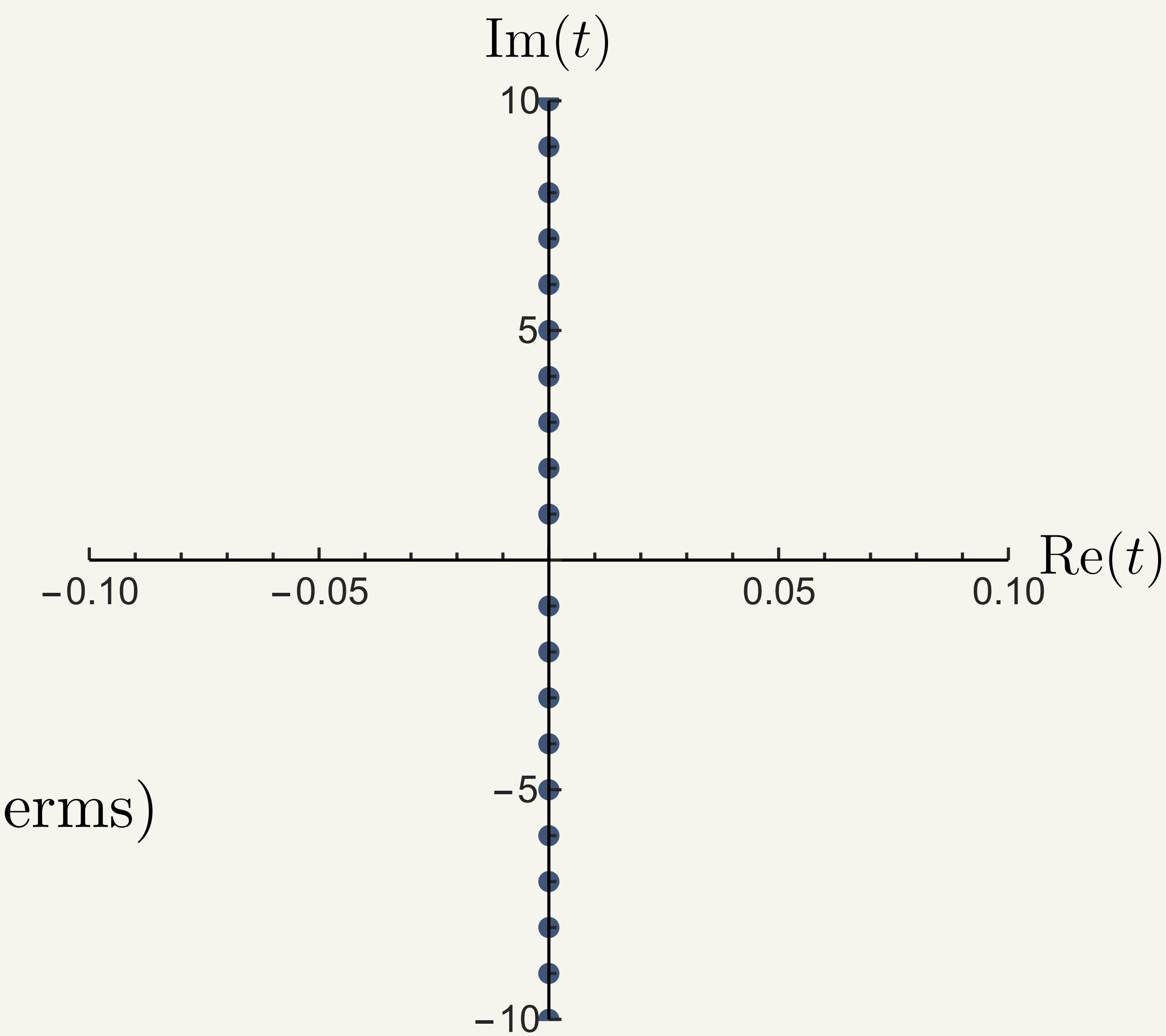
- The existence of repeated branch points is obscured.

Two-loop analysis

- Compare to one-loop Borel singularity structure

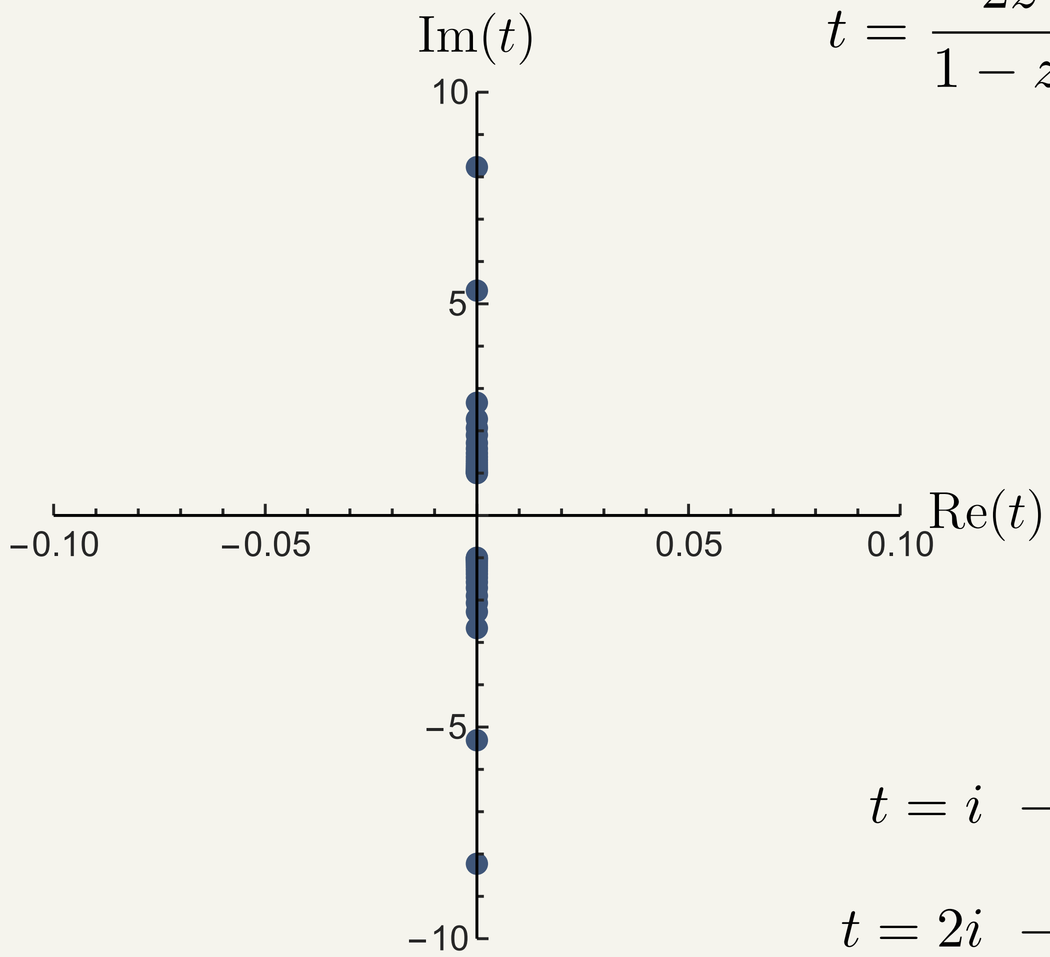


(50 terms)



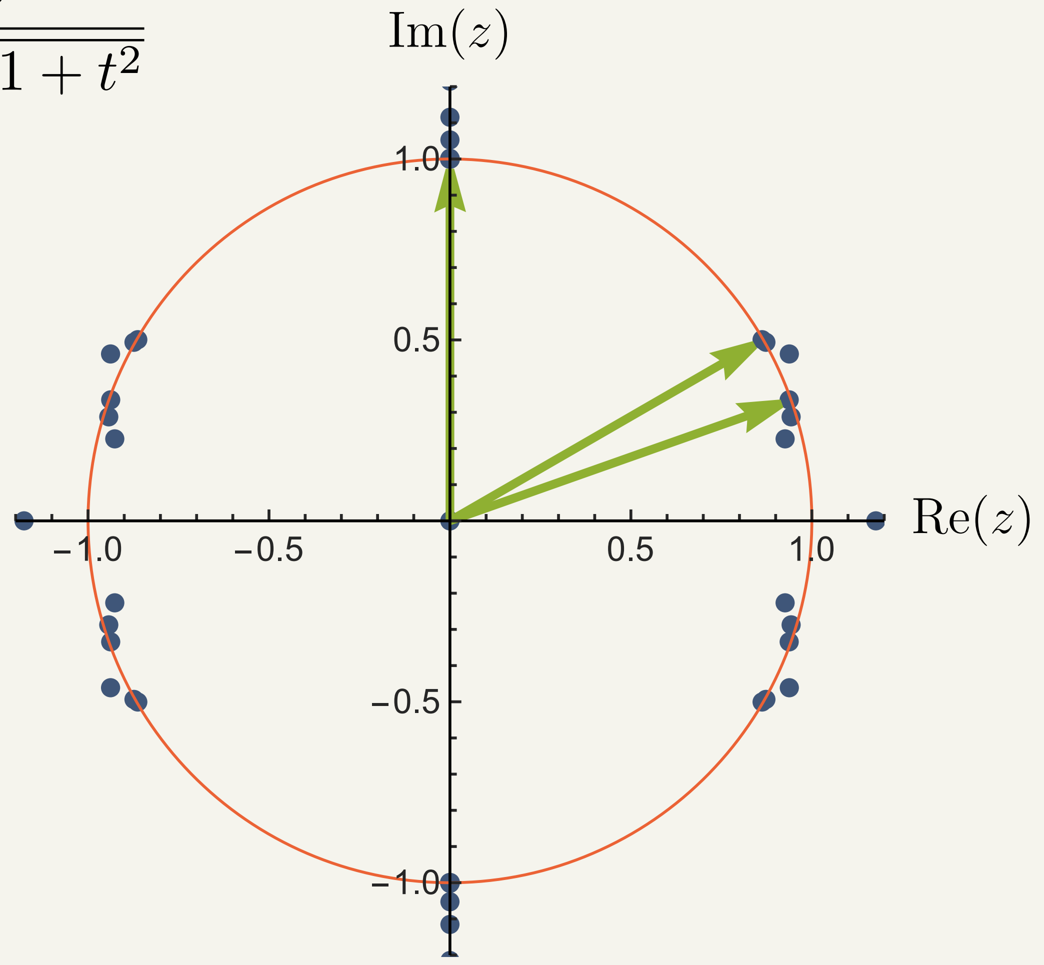
Two-loop analysis

- Exponential terms suppressed relative to power-law corrections
- Access them directly $\Rightarrow$ conformal map



$$t = \frac{2z}{1-z^2}, \qquad z = \frac{t}{1+\sqrt{1+t^2}}$$

$$\Rightarrow$$



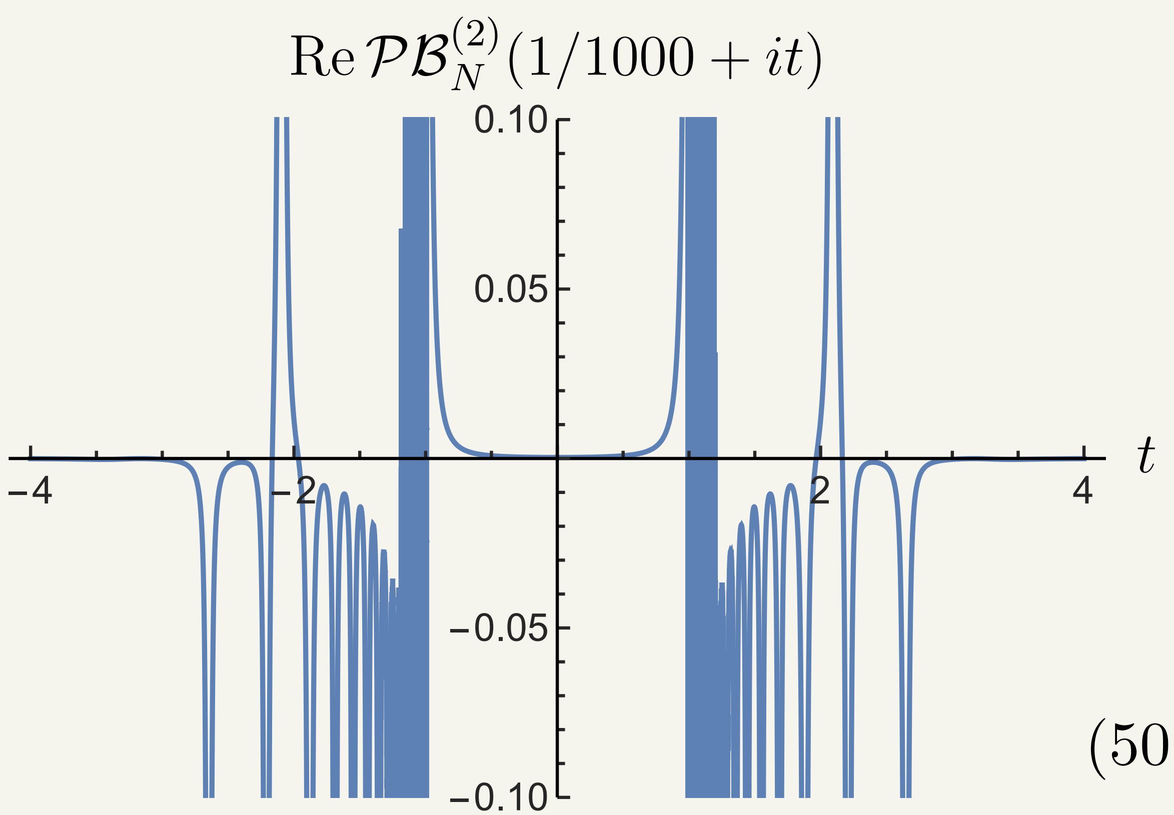
$$t = i \;\rightarrow\; \arg z = \pm\frac{\pi}{2}$$

$$t = 2i \;\rightarrow\; \arg z = \pm\frac{\pi}{6}$$

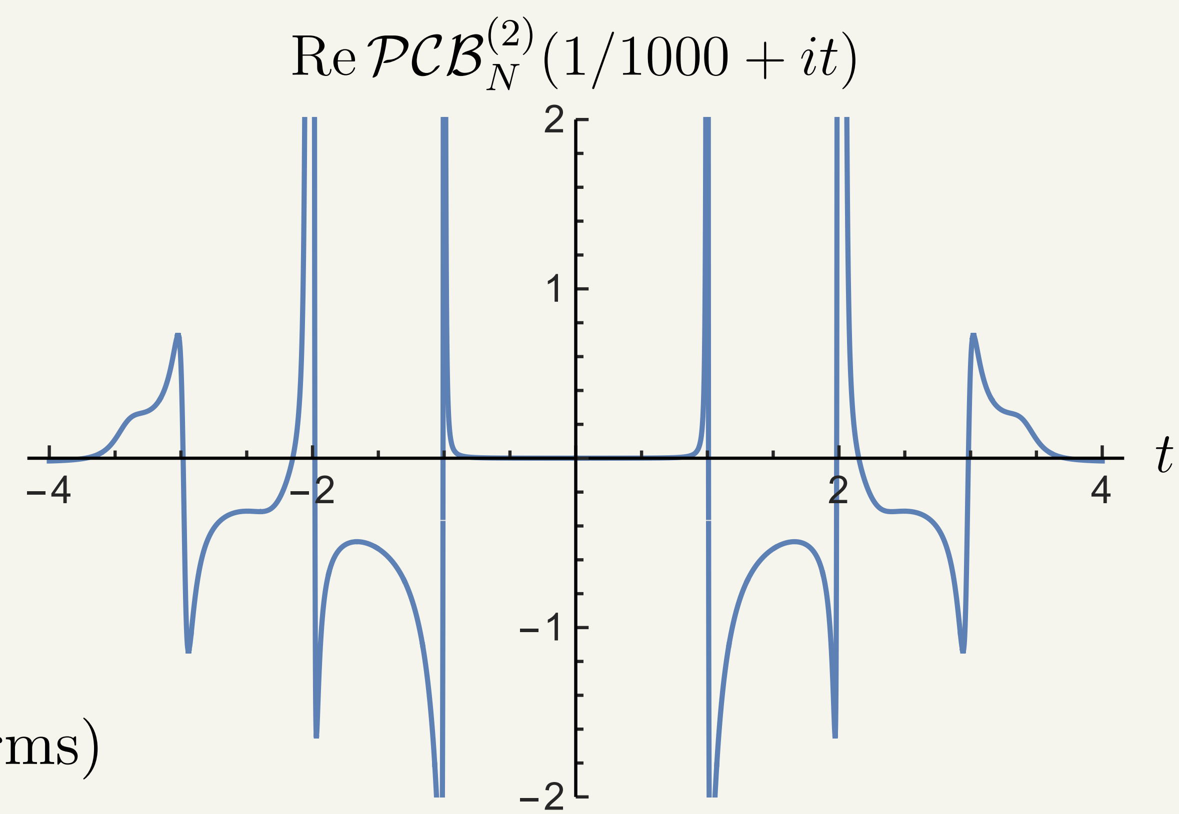
$$t = 3i \;\rightarrow\; \arg z = \pm\arctan\frac{1}{2\sqrt{2}}$$

Two-loop analysis

- Algorithm:
  - Perform the conformal map on the Borel transform function.
  - Series expand to the same order as in the Borel variable.
  - Construct a Padé approximant.
  - Apply the inverse conformal map to return to the Borel plane.



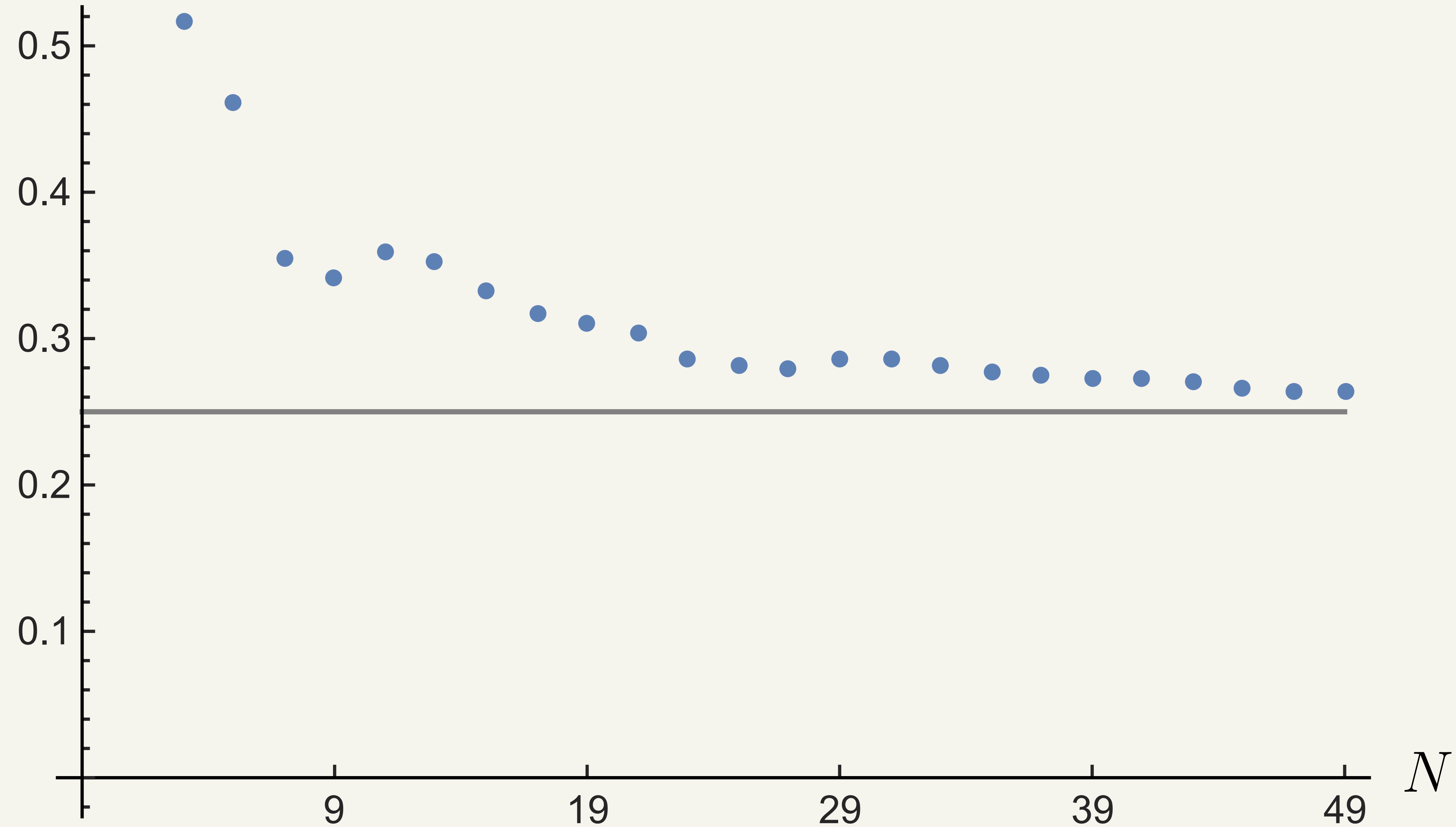
(50 terms)



Two-loop analysis

- From encoding the strong field behavior, we recover the second beta function coefficient

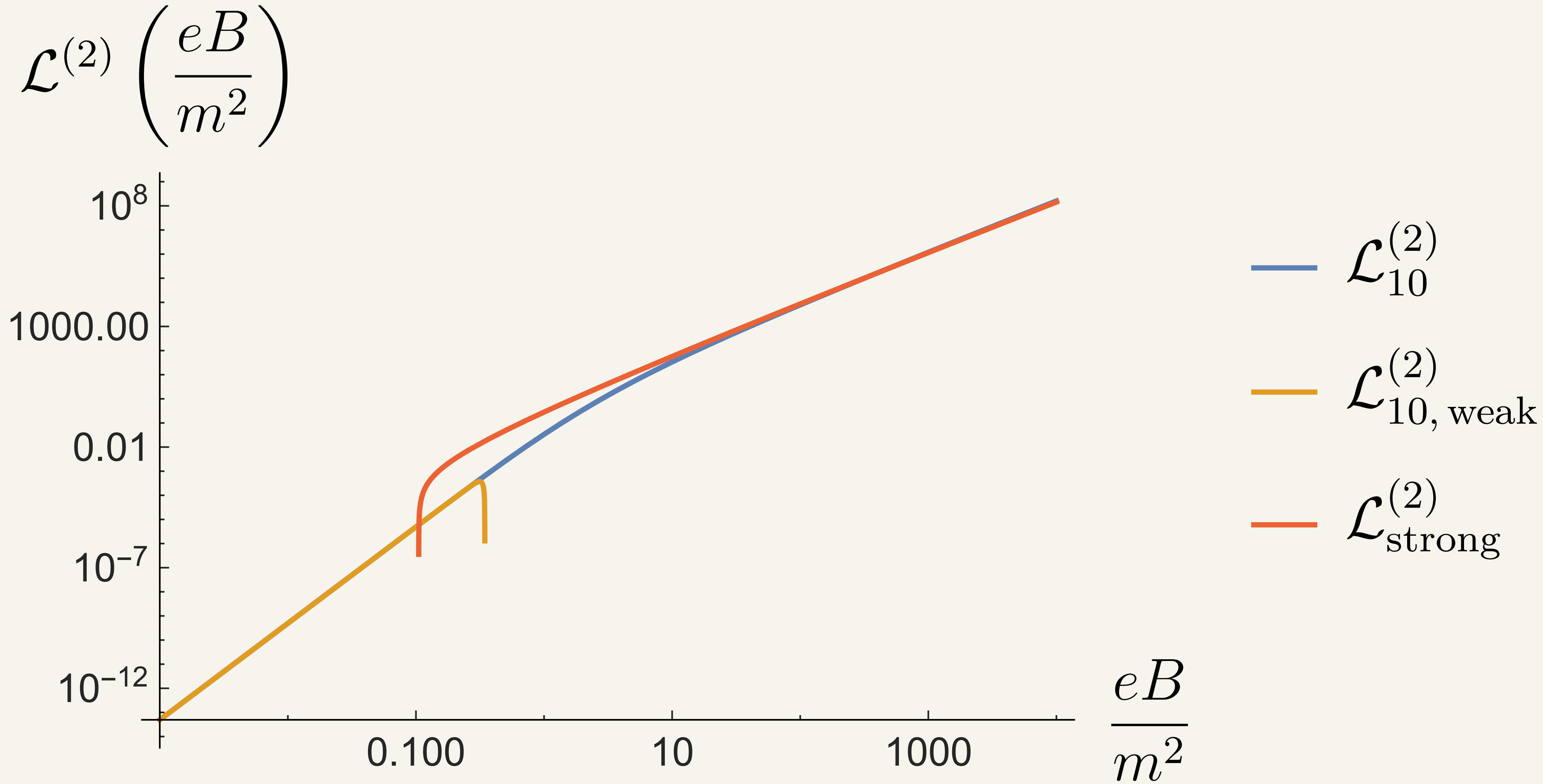
$$\pi t \cdot \mathcal{PB}_N^{(2)}(t)\Big|_{t\to\infty}$$



Two-loop analysis

- Performing the Borel sum, we can extrapolate from weak magnetic field to strong magnetic field (10 terms)

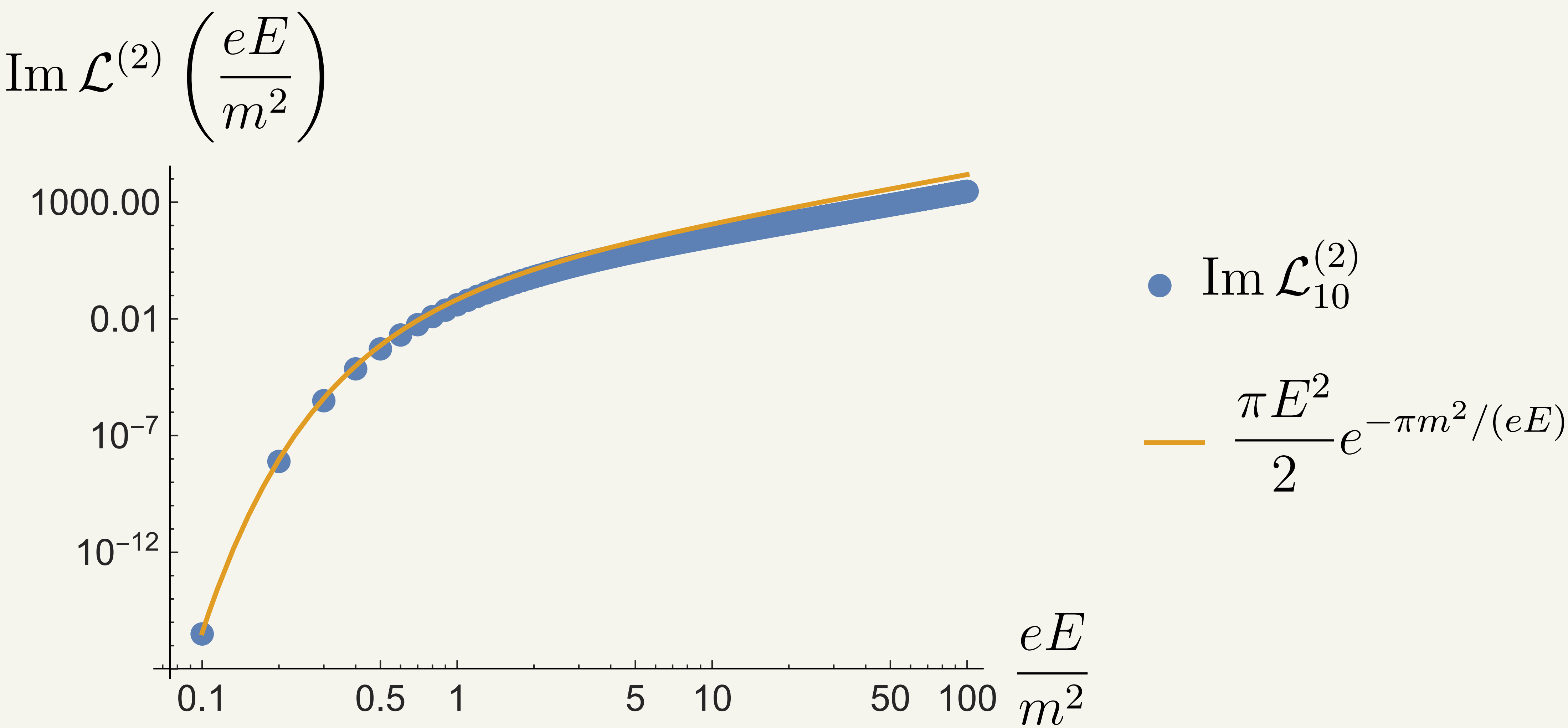
$$\mathcal{L}_N^{(2)}\left(\frac{eB}{m^2}\right) = \frac{\pi B^2}{2}\int_0^\infty \mathrm{d}t\, \mathcal{PB}_N^{(2)}(t) e^{-\pi m^2 t/(eB)}$$



Two-loop analysis

- Analytic continuation to constant electric field (10 terms)

$$\mathcal{L}_N^{(2)}\left(\frac{eE}{m^2}\right) = -\frac{\pi E^2}{2}\int_0^{e^{\pm i\varepsilon}\infty} i\,\mathrm{d}t\,\mathcal{P}\mathcal{B}_N^{(2)}(it)e^{-\pi m^2 t/(eE)}$$



Sub-leading corrections

- How is the leading growth of the coefficients the same but the Borel singularity structure so different?

- Resolution: superposition of pole and branch point

$$a_n^{(2)} - (-1)^n \frac{\Gamma(2n+2)}{\pi^{2n+2}} \approx (1.65) \times (-1)^n \frac{\Gamma(2n+\frac{5}{4})}{\pi^{2n+2}}$$

- This implies a contribution to the imaginary part of the Borel sum

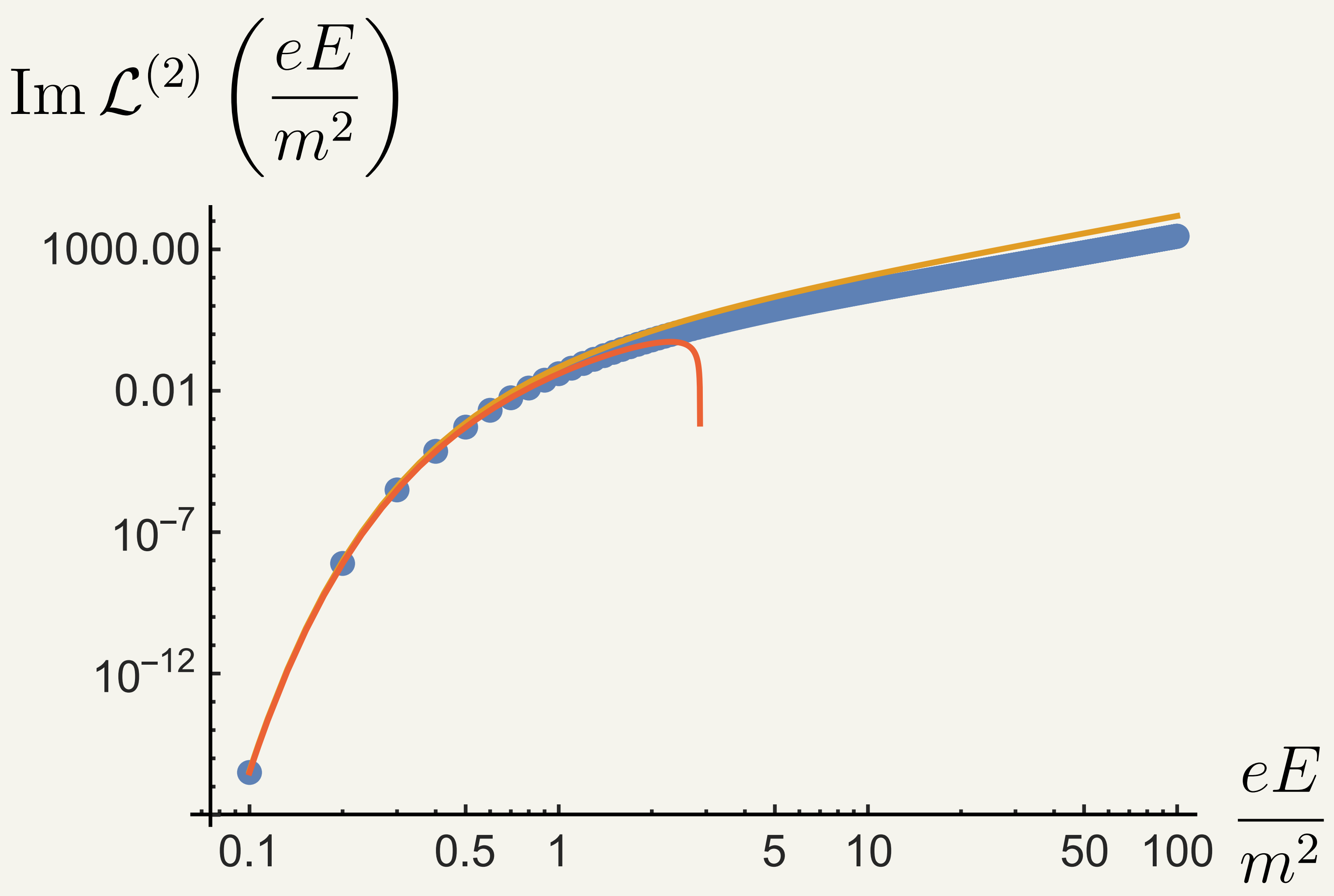
$$\operatorname{Im} \mathcal{L}^{(2)}\left(\frac{eE}{m^2}\right) \sim \frac{\pi E^2}{2} e^{-\pi m^2/(eE)} \left(1 - 1.65 \left(\frac{eE}{\pi m^2}\right)^{3/4} + \dots\right) + \dots$$

Novel result at two-loop

- We conjecture power law fluctuations of the form $\left(\frac{eE}{\pi m^2}\right)^{3n/4}$, $\quad n \in \mathbb{Z}^+$

Sub-leading corrections

- Including sub-leading corrections



$$\mathrm{Im}\,\mathcal{L}^{(2)}\left(\frac{eE}{m^2}\right) \sim \frac{\pi E^2}{2} e^{-\pi m^2/(eE)} \left(1 - 1.65\left(\frac{eE}{\pi m^2}\right)^{3/4} + 2.42\left(\frac{eE}{\pi m^2}\right)^{6/4} - 1.92\left(\frac{eE}{\pi m^2}\right)^{9/4} + \dots\right) + \dots$$

Sub-leading corrections

- What about higher-order instantons?

- Accessing the leading contribution from the second instanton should be possible through the conformal map.

- This would give quantitative evidence either for or against the conjectured form of higher instanton contributions

$$\operatorname{Im}\mathcal{L}^{(2)}\left(\frac{eE}{m^2}\right) \sim \frac{\pi E^2}{2}\left\{e^{-\pi m^2/(eE)}\left(1+\dots\right)+\sqrt{\frac{m^2}{eE}}\sum_{k=2}^{\infty}e^{-k\pi m^2/(eE)}\left(-c_k+\sqrt{\frac{eE}{m^2}}+\dots\right)\right\}$$

✓ ?

$$\operatorname{Im}\mathcal{L}^{(2)}\left(\frac{eE}{m^2}\right) \sim \frac{\pi E^2}{2}\left\{e^{-\pi m^2/(eE)}\sum_{n=0}^{\infty}b_n\left(\frac{eE}{\pi m^2}\right)^{3n/4}+\sqrt{\frac{m^2}{eE}}\sum_{k=2}^{\infty}e^{-k\pi m^2/(eE)}\left(-c_k+\sqrt{\frac{eE}{m^2}}+\dots\right)\right\}$$

Two-loop analysis

- Summary:
  - Using only the first 10 terms in the weak magnetic field expansion, we were able to:
    - Extrapolate from weak magnetic field to strong magnetic field over eight orders of magnitude (no simple way to plot exact expression!).
    - Analytically continue to a constant electric field and determine the form of the power-law fluctuations about the one instanton contribution, leading to an updated conjecture on the structure of the pair production rate at two-loop order.
  - Using the first 50 terms, we were able to:
    - Recover the second beta function coefficient by encoding the strong field behavior into the Padé approximation.
    - Resolve the exponentially suppressed second and third instanton contributions for an electric background field, using a conformal map.

Overview

- Resurgence and Borel summation
- Euler-Heisenberg effective action
- One-loop analysis
- Two-loop analysis
- Future work and conclusion

Future work

- Investigate higher order instanton contributions for constant background field configurations.

- Apply Borel resummation to the case of an inhomogeneous background field at one-loop $\Rightarrow$ Borel transform known in closed form

$$\vec{B}(x) = \vec{B}\,\mathrm{sech}^2\left(\frac{x}{\lambda}\right), \quad \vec{E}(t) = \vec{E}\,\mathrm{sech}^2\left(\omega t\right)$$

- Study the structure of higher loops (exact results for three-loop only known in 1+1 dim., but can generate perturbative expansion)

$$\mathcal{L}_{\mathrm{EH}}\left(\alpha, \frac{eB}{m^2}\right) \sim \sum_{\ell=1}^{\infty} \left(\frac{\alpha}{\pi}\right)^{\ell} \mathcal{L}^{(\ell)}\left(\frac{eB}{m^2}\right)$$

$eB \ll m^2$:

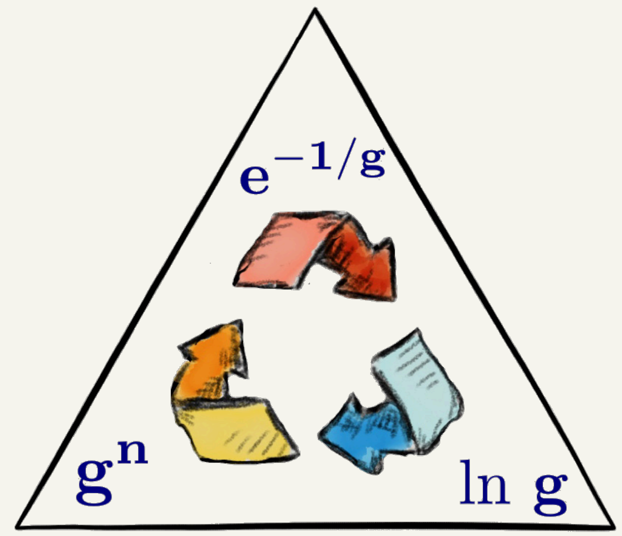
$$\mathcal{L}^{(\ell)}\left(\frac{eB}{m^2}\right) \sim \frac{\pi^{2(\ell-2)} B^2}{(\ell-1)!}\left(\frac{eB}{m^2}\right)^2 \sum_{n=0}^{\infty} a_n^{(\ell)} \left(\frac{eB}{m^2}\right)^{2n}$$

$eB \gg m^2$:

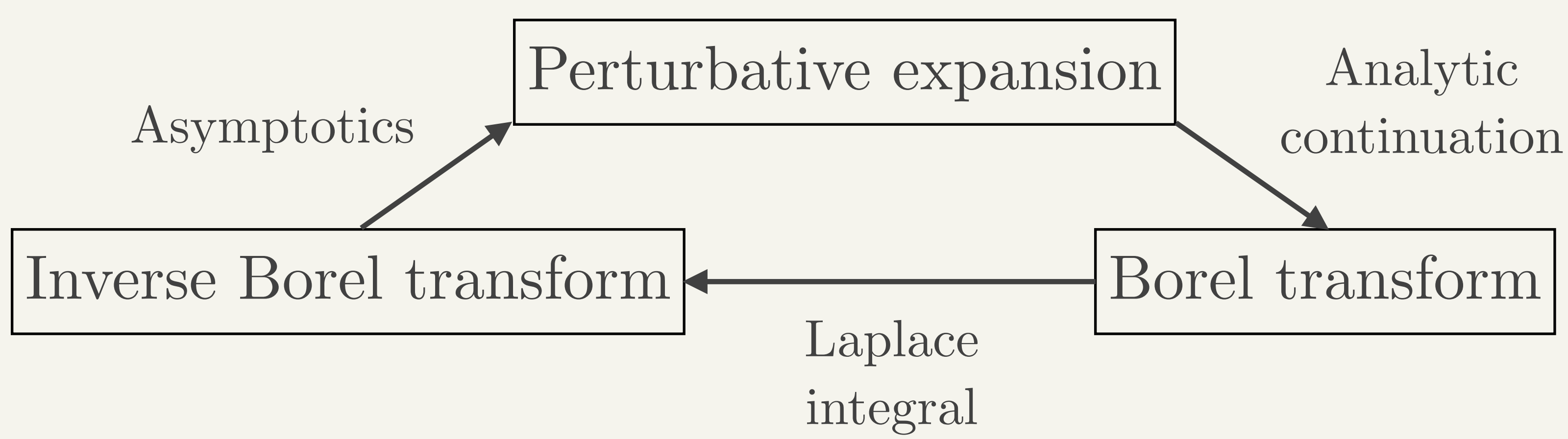
$$\mathcal{L}^{(\ell)}\left(\frac{eB}{m^2}\right) \sim \frac{\beta_2}{\beta_1^2}(\beta_1)^{\ell} \frac{B^2}{2(\ell-1)} \ln^{\ell-1}\left(\frac{eB}{\pi m^2}\right), \ \ell \geq 2$$

Conclusion

- Resurgence: unification of perturbative and non-perturbative physics



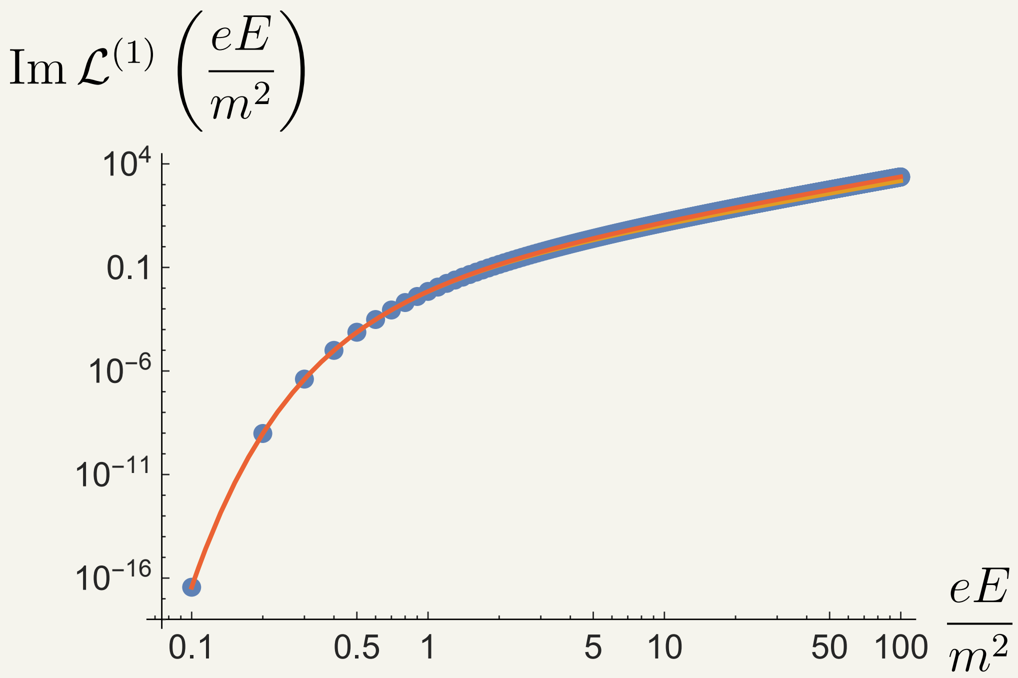
- Borel summation is a powerful methodology through which formally divergent asymptotic series can be resummed to yield a well-defined analytic continuation sensitive to non-perturbative, imaginary contributions.

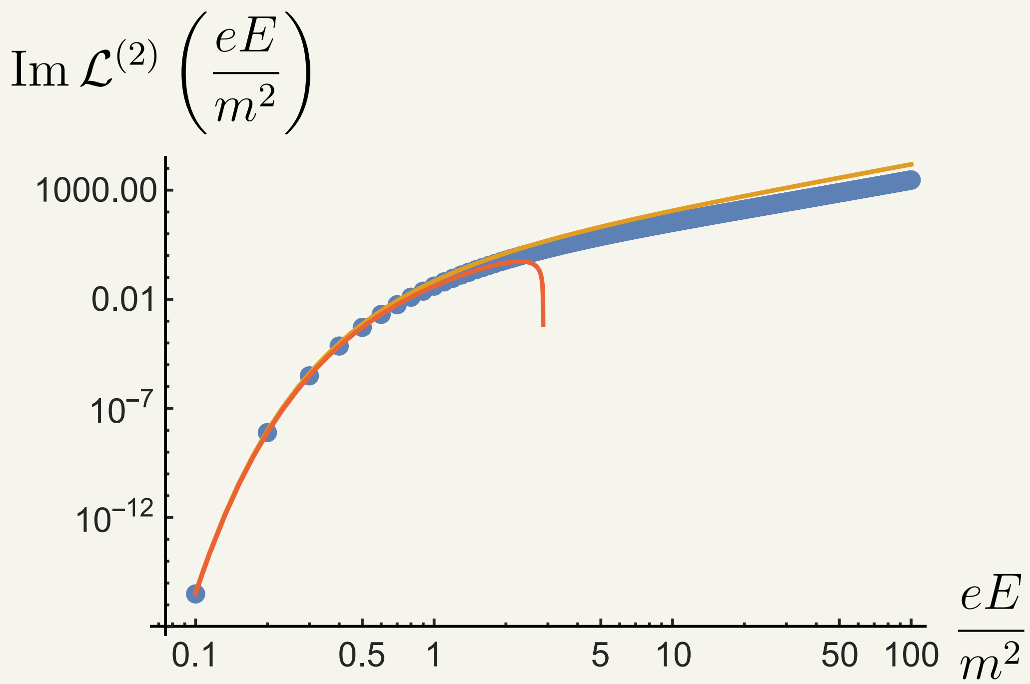


Conclusion

- The Euler-Heisenberg effective action at one and two loop order demonstrates, as a proof-of-principle, the applicability of Borel summation to physical systems.

- From a finite amount of magnetic field perturbative data as an input, we were able to accurately reconstruct both the strong magnetic field limit and the non-perturbative pair production rate for a background electric field.





- This led to a novel resolution of the power-law fluctuations about the one-instanton contribution at two loop order.

- This technique has clear applicability at higher loop orders and other related systems.

Thank you

Questions?

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- Disclaimer: any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

Literature

- G. Dunne, Z. Harris, "Higher-loop Euler-Heisenberg transseries structure," *Phys. Rev. D*, **103**, 065015, (2021).

- O. Costin, G. Dunne, "Physical Resurgent Extrapolation" *Phys. Lett. B* **808**, 135627, (2020).

- F. Karbstein, "All-Loop Result for the Strong Magnetic Field Limit of the Heisenberg-Euler Effective Lagrangian," *Phys. Rev. Lett.* **122**, 211602, (2019).

- S. Lebedev, V. Ritus, "Virial representation of the imaginary part of the Lagrange function of the electromagnetic field," *Sov. Phys. JETP* **59**, 237, (1984).

- W. Heisenberg, H. Euler, "Consequences of Dirac's Theory of the Positron," *Z. Phys.* **98**, 714, (1936).