Chapter 24

Field Integrals

24.1

The Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2}m^2 A^2$$

can be rewritten by integrating by parts. Since the field $A_\mu(x)$ is assumed to vanish as $x \to \infty$, we can throw away the boundary term without changing the equations of motion. Therefore, we have

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\big(A_\nu \partial^2 A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu\big) + \frac{1}{2}m^2 A_\mu A^\mu \\
&= \frac{1}{2}\big(A^\nu g_{\mu\nu}\partial^2 A^\mu - A^\nu \partial_\mu \partial_\nu A^\mu\big) + \frac{1}{2}A^\nu g_{\mu\nu} m^2 A^\mu \\
&= \frac{1}{2}A^\nu \Big[(\partial^2 + m^2) g_{\mu\nu} - \partial_\mu \partial_\nu\Big] A^\mu \\
&\boxed{\mathcal{L} = \frac{1}{2}A^\nu \hat{K}_{\mu\nu} A^\mu}
\end{aligned}$$

where

$$\hat{K}_{\mu\nu} = (\partial^2 + m^2) g_{\mu\nu} - \partial_\mu \partial_\nu$$

We can also check that

$$\tilde{G}_{0\nu\lambda}(p) = \frac{-i}{p^2 - m^2}\Big(g_{\nu\lambda} - \frac{p_\nu p_\lambda}{m^2}\Big)$$

indeed satisfies

$$\Big[-(p^2 - m^2) g^{\mu\nu} + p^\mu p^\nu\Big]\tilde{G}_{0\nu\lambda}(p) = i g^\mu{}_\lambda$$

just by direct substitution

$$\begin{aligned}
\Big[-(p^2 - m^2) g^{\mu\nu} + p^\mu p^\nu\Big]\tilde{G}_{0\nu\lambda}(p) &= \Big[-(p^2 - m^2) g^{\mu\nu} + p^\mu p^\nu\Big]\frac{-i}{p^2 - m^2}\Big(g_{\nu\lambda} - \frac{p_\nu p_\lambda}{m^2}\Big) \\
&= i\Big(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2 - m^2}\Big)\Big(g_{\nu\lambda} - \frac{p_\nu p_\lambda}{m^2}\Big) \\
&= i\Big(g^\mu{}_\lambda - \frac{p^\mu p_\lambda}{m^2} - \frac{p^\mu p_\lambda}{p^2 - m^2} + \frac{p^2 p^\mu p_\lambda}{m^2(p^2 - m^2)}\Big) \\
&= i\Big(g^\mu{}_\lambda - p^\mu p_\lambda\Big(\frac{p^2}{m^2(p^2 - m^2)} - \frac{p^2}{m^2(p^2 - m^2)}\Big)\Big)
\end{aligned}$$

$$\boxed{\Big[-(p^2 - m^2) g^{\mu\nu} + p^\mu p^\nu\Big]\tilde{G}_{0\nu\lambda}(p) = i g^\mu{}_\lambda}$$

24.2

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{8}\phi^4$$

We can introduce a new field $\sigma(x)$ and define a new Lagrangian

$$\mathcal{L}' = \mathcal{L} + \frac{1}{2g}\left(\sigma - \frac{g}{2}\phi^2\right)^2 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{2g}\sigma^2 - \frac{1}{2}\sigma\phi^2$$

The dynamics are contained within the Green's functions for the theory, which we can access via the generating functional

$$\mathcal{Z}[J] = \frac{\displaystyle\int \mathcal{D}\phi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^4x \left(\mathcal{L}' + J\phi\right)\right]}{\displaystyle\int \mathcal{D}\phi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^4x\, \mathcal{L}'\right]}$$

Since the integral over the σ field is quadratic, we can "integrate out" this field and study the effective (in this case they're exact) dynamics of the ϕ field

$$\begin{aligned}
\mathcal{Z}[J] &= \frac{\displaystyle\int \mathcal{D}\phi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{2g}\sigma^2 - \frac{1}{2}\sigma\phi^2 + J\phi\right)\right]}{\displaystyle\int \mathcal{D}\phi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{2g}\sigma^2 - \frac{1}{2}\sigma\phi^2\right)\right]} \\
&= \frac{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi\right)\right]\int \mathcal{D}\sigma \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2g}\sigma^2 - \frac{1}{2}\phi^2\sigma\right)\right]}{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2\right)\right]\int \mathcal{D}\sigma \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2g}\sigma^2 - \frac{1}{2}\phi^2\sigma\right)\right]} \\
&= \frac{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + J\phi\right)\right]\exp\left[-\frac{i}{2}\int \mathrm{d}^4x\, g\left(-\frac{\phi^2}{2}\right)^2\right]}{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2\right)\right]\exp\left[-\frac{i}{2}\int \mathrm{d}^4x\, g\left(-\frac{\phi^2}{2}\right)^2\right]} \\
&= \frac{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{8}\phi^4 + J\phi\right)\right]}{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{8}\phi^4\right)\right]}
\end{aligned}$$

$$\boxed{\mathcal{Z}[J] = \frac{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x \left(\mathcal{L} + J\phi\right)\right]}{\displaystyle\int \mathcal{D}\phi \exp\left[i\int \mathrm{d}^4x\, \mathcal{L}\right]}}$$

As we can see, the generating functional for the $\sigma\phi^2$ theory is identical to that of the ϕ^4 theory, showing that the σ field has no effects on the dynamics of the system. We can separately confirm this by looking at the equation of motion for the σ field

$$\frac{\partial \mathcal{L}'}{\partial \sigma} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \sigma)} = 0$$

$$\boxed{\sigma = \frac{g}{2}\phi^2}$$

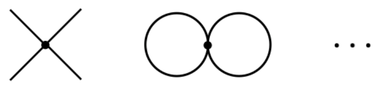
There are no time derivatives in the equation of motion, which confirms that the field σ does not contribute to the dynamics of the system. If we promote the fields to operators, we can consider a vacuum-to-vacuum expansion of the S operator

$$\begin{aligned}
\langle 0|S|0\rangle &= \langle 0|T\left[\exp\left(-i\int \mathrm{d}^4x\, \frac{1}{2}\sigma\phi^2\right)\right]|0\rangle \\
\langle 0|S-1|0\rangle &= \frac{-i}{2}\int \mathrm{d}^4x\, \langle 0|T\left[\sigma_x\phi_x\phi_x\right]|0\rangle + \frac{1}{2!}\left(\frac{-i}{2}\right)^2\int \mathrm{d}^4x\, \mathrm{d}^4y\, \langle 0|T\left[\sigma_x\phi_x\phi_x\sigma_y\phi_y\phi_y\right]|0\rangle + \ldots
\end{aligned}$$

We can associate contractions of the ϕ field with the standard free field propagator, diagrammatically as a line connecting two spacetime points. Since the σ field isn't dynamical, nothing can propagate through it. Therefore, contractions of the σ field should be associated with taking two spacetime points and identifying them (diagrammatically as two dots on top of each other). With this, we can draw the first few Feynman diagrams

$$\sigma_x\sigma_y \langle 0|T\left[\phi_x\phi_x\phi_y\phi_y\right]|0\rangle\,,\ \sigma_x\sigma_y\phi_x\phi_y\phi_x\phi_y,\ \ldots$$

as



We now see that contractions on the σ field play the role of the interaction vertex in the ϕ^4 theory, and indeed the dynamics of the two theories are the same.

24.3

Consider the one-dimensional integral

$$Z(J) = \int \mathrm{d}x\, e^{-\frac{1}{2}Ax^2 - \frac{\lambda}{4!}x^4 + Jx}$$

We can separate and expand the interaction term in a Taylor series, which allows us to write

$$\begin{aligned} Z(J) &= \int \mathrm{d}x\, e^{-\frac{\lambda}{4!}x^4} e^{-\frac{1}{2}Ax^2+Jx} \\ &= \int \mathrm{d}x \left(1 - \frac{\lambda}{4!}x^4 + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 x^8 + \ldots\right) e^{-\frac{1}{2}Ax^2+Jx} \\ &= \int \mathrm{d}x \left(1 - \frac{\lambda}{4!}\frac{\partial^4}{\partial J^4} + \frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2 \frac{\partial^8}{\partial J^8} + \ldots\right) e^{-\frac{1}{2}Ax^2+Jx} \end{aligned}$$

$$\boxed{Z(J) = \sum_{n=0}^{\infty}\left[\frac{1}{n!}\left(-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}\right)^n\right]\int \mathrm{d}x\, e^{-\frac{1}{2}Ax^2+Jx}} \tag{24.1}$$

This differential operator can be rewritten as

$$Z(J) = e^{-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}} \int \mathrm{d}x\, e^{-\frac{1}{2}Ax^2+Jx}$$

where we've completely removed the interaction term from the integral. All that remains is a one-dimensional Gaussian integral, which is easily evaluated

$$\boxed{Z(J) = e^{-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}}\left[\left(\frac{2\pi}{A}\right)^{1/2} e^{\frac{J^2}{2A}}\right]} \tag{24.2}$$

Identifying the term in square brackets as the free generating function $Z_0(J)$, we see that we can obtain the full generating functional by acting this exponentiated differential operator on the free generating function

$$Z(J) = e^{-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}} Z_0(J)$$

24.4

In analogy with the previous problem, the full generating functional for a ϕ^4 theory is given by

$$Z[J] = \exp\left(-\frac{i\lambda}{4!}\int \mathrm{d}^4z\,\frac{\delta^4}{\delta J(z)^4}\right)\mathcal{Z}_0[J]$$

where $\mathcal{Z}_0[J]$ is the normalized generating functional for a free scalar field theory

$$\mathcal{Z}_0[J] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)\right)$$

and $\Delta(x-y) = \Delta(y-x)$ is the Feynman propagator for a scalar field

$$-(\partial^2 + m^2)\Delta(x-y) = i\delta^{(4)}(x-y)$$

The first order term in this expansion is given by

$$Z_1[J] = \left[-\frac{i\lambda}{4!}\int \mathrm{d}^4z\,\frac{\delta^4}{\delta J(z)^4}\right]\mathcal{Z}_0[J]$$

Using the definition of functional differentiation, we can calculate the derivatives as follows

$$\begin{aligned}
\frac{\delta\mathcal{Z}_0[J]}{\delta J(z)} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{Z}_0[J+\varepsilon\delta(z-x)]\bigg|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,(J(x)+\varepsilon\delta(z-x))\Delta(x-y)(J(y)+\varepsilon\delta(z-y))\right)\bigg|_{\varepsilon=0} \\
&= \left(-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big[J(x)\Delta(x-y)\delta(z-y) + J(y)\Delta(x-y)\delta(z-x)\Big]\right)\mathcal{Z}_0[J] \\
&= \left(-\frac{1}{2}\int \mathrm{d}^4x\,J(x)\Delta(x-z) - \frac{1}{2}\int \mathrm{d}^4y\,J(y)\Delta(z-y)\right)\mathcal{Z}_0[J]
\end{aligned}$$

$$\boxed{\frac{\delta\mathcal{Z}_0[J]}{\delta J(z)} = \left[-\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right]\mathcal{Z}_0[J]} \tag{24.3}$$

$$\begin{aligned}
\frac{\delta^2\mathcal{Z}_0[J]}{\delta J(z)^2} &= \frac{\delta}{\delta J(z)}\left[-\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right]\mathcal{Z}_0[J] + \left[-\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right]\frac{\delta\mathcal{Z}_0[J]}{\delta J(z)} \\
&= -\Delta(z-z)\mathcal{Z}_0[J] + \left[-\int \mathrm{d}^4x\,\Delta(z-x)J(x)\right]\left[-\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right]\mathcal{Z}_0[J]
\end{aligned}$$

$$\boxed{\frac{\delta^2\mathcal{Z}_0[J]}{\delta J(z)^2} = \left[-\Delta(z-z) + \left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^2\right]\mathcal{Z}_0[J]} \tag{24.4}$$

$$\begin{aligned}
\frac{\delta^3\mathcal{Z}_0[J]}{\delta J(z)^3} = 2\Delta(z-z)\left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}\mathcal{Z}_0[J] + \Bigg[-\Delta(z-z) \\
+ \left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^2\Bigg]\frac{\delta\mathcal{Z}_0[J]}{\delta J(z)}
\end{aligned}$$

$$\boxed{\frac{\delta^3\mathcal{Z}_0[J]}{\delta J(z)^3} = \left[3\Delta(z-z)\left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\} - \left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^3\right]\mathcal{Z}_0[J]} \tag{24.5}$$

$$\begin{aligned}
\frac{\delta^4\mathcal{Z}_0[J]}{\delta J(z)^4} &= \left[3\Delta(z-z)^2 - 3\Delta(z-z)\left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^2\right]\mathcal{Z}_0[J] \\
&+ \left[3\Delta(z-z)\left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\} - \left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^3\right]\frac{\delta\mathcal{Z}_0[J]}{\delta J(z)}
\end{aligned}$$

$$\boxed{\frac{\delta^4\mathcal{Z}_0[J]}{\delta J(z)^4} = \left[3\Delta(z-z)^2 - 6\Delta(z-z)\left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^2 + \left\{\int \mathrm{d}^4y\,\Delta(z-y)J(y)\right\}^4\right]\mathcal{Z}_0[J]} \tag{24.6}$$

Plugging in this expression

$$Z_1[J] = -\frac{i\lambda}{4!}\int \mathrm{d}^4 z \left[3\Delta(z-z)^2 - 6\Delta(z-z)\left\{\int \mathrm{d}^4 y\, \Delta(z-y)J(y)\right\}^2 + \left\{\int \mathrm{d}^4 y\, \Delta(z-y)J(y)\right\}^4\right]\mathcal{Z}_0[J]$$

we can expand out the integrals to write

$$\boxed{\begin{aligned} Z_1[J] = -i\lambda\Bigg[&\frac{1}{8}\int \mathrm{d}^4 z\, \Delta(z-z)^2 - \frac{1}{4}\int \mathrm{d}^4 z\, \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2\, \Delta(z-z)J(y_1)\Delta(z-y_1)J(y_2)\Delta(z-y_2) \\ &+ \frac{1}{4!}\int \mathrm{d}^4 z\, \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2\, \mathrm{d}^4 y_3\, \mathrm{d}^4 y_4\, \Delta(z-y_1)J(y_1)\Delta(z-y_2)J(y_2) \\ &\times \Delta(z-y_3)J(y_3)\Delta(z-y_4)J(y_4)\Bigg]\mathcal{Z}_0[J] \end{aligned}} \tag{24.7}$$

These terms of $\mathcal{O}(J) = 0, 2,$ and 4 can be expressed as the Feynman diagrams shown below