Chapter 23

Path integrals: I said to him, ‘You’re crazy’

23.1

For the Lagrangian

$$L = \frac{1}{2}x\hat{\mathcal{A}}x + bx$$

assuming x is an implicit function of t, the equations of motion yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \hat{\mathcal{A}}x + b &= 0 \\ x &= -\hat{\mathcal{A}}^{-1}b\end{aligned}$$

Plugging this in, the Lagrangian evaluated along the equation of motion is given by

$$\begin{aligned}L_{\mathrm{EOM}} &= \frac{1}{2}\left(-\hat{\mathcal{A}}^{-1}b\right)\hat{\mathcal{A}}\left(-\hat{\mathcal{A}}^{-1}b\right) + b\left(-\hat{\mathcal{A}}^{-1}b\right) \\ &= \frac{1}{2}b\hat{\mathcal{A}}^{-1}b - b\hat{\mathcal{A}}^{-1}b \\ \Aboxed{L_{\mathrm{EOM}} &= -b\frac{1}{2\hat{\mathcal{A}}}b}\end{aligned}$$

Since this is the expression which appears in the evaluation of the Gaussian integral

$$\int \mathcal{D}[x]\exp\left(\frac{i}{2}x\hat{\mathcal{A}}x + bx\right) = \frac{B}{\sqrt{\det\hat{\mathcal{A}}}}\exp\left(-ib\frac{1}{2\hat{\mathcal{A}}}b\right)$$

we see that the exponential is just the classical action evaluated along the classical equation of motion

$$\int \mathcal{D}[x]\exp\left(\frac{i}{2}x\hat{\mathcal{A}}x + bx\right) = \frac{B}{\sqrt{\det\hat{\mathcal{A}}}}\exp\left(iS_{\mathrm{EOM}}\right)$$

23.2

The second moment of the Gaussian distribution

$$\mu_2 = \int_{-\infty}^{\infty}\mathrm{d}x\, x^2 e^{-\frac{1}{2}ax^2}$$

can be solved for using the following trick. Since the zeroth moment of the inhomogeneous Gaussian distribution is given by

$$\mu_{0,b} = \int_{-\infty}^{\infty}\mathrm{d}x\, e^{-\frac{1}{2}ax^2 + bx} = \sqrt{\frac{2\pi}{a}}e^{b^2/2a}$$

we can write the homogeneous second moment as

$$\mu_2 = \frac{\partial^2}{\partial b^2}(\mu_{0,b})\bigg|_{b=0}$$

Using the result of the zeroth moment, we find

$$\boxed{\mu_2 = \sqrt{\frac{2\pi}{a}}\frac{1}{a}}$$

Denoting the n-th normalized moment as $\langle x^n \rangle$

$$\langle x^n \rangle = \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, x^n e^{-\frac{1}{2}ax^2}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2}}$$

we can work out the first two non-zero moments as follows

$$\begin{aligned}
\langle x^2 \rangle &= \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, x^2 e^{-\frac{1}{2}ax^2}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2}} \\
&= \sqrt{\frac{a}{2\pi}}\frac{\partial^2}{\partial b^2}(\mu_{0,b})\bigg|_{b=0}
\end{aligned}$$

$$\boxed{\langle x^2 \rangle = \frac{1}{a}}$$

$$\begin{aligned}
\langle x^4 \rangle &= \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, x^4 e^{-\frac{1}{2}ax^2}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2}} \\
&= \sqrt{\frac{a}{2\pi}}\frac{\partial^4}{\partial b^4}(\mu_{0,b})\bigg|_{b=0}
\end{aligned}$$

$$\boxed{\langle x^4 \rangle = \frac{3}{a^2}}$$

In order to really see the pattern, we can go one step further and find that $\langle x^6 \rangle = 15/a^3$, at which point we can extrapolate and write

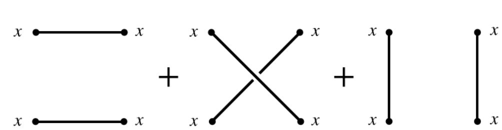
$$\boxed{\langle x^n \rangle = \begin{cases} 0 & n \text{ odd} \\ \dfrac{(n-1)!!}{a^{n/2}} & n \text{ even} \end{cases}}$$

which can be checked and is indeed correct. This can be understood diagrammatically by associating each factor of $1/a$ with a line which connects two factors of x. For $\langle x^2 \rangle$, there are only two factors of x, and thus only one way to connect them

$$\langle x^2 \rangle =$$

As for $\langle x^4 \rangle$, there are three ways to combine four factors of x into pairs

$$\langle x^4 \rangle =$$



Therefore, for n even we have

$$\boxed{\langle x^n \rangle = \sum_{\text{All graphs}} \Big(n \text{ points connected into pairs} \Big)}$$

Upgrading to the N-dimensional Gaussian integral

$$\mathcal{K} = \int_{-\infty}^{\infty} \mathrm{d}x_1 \dots \mathrm{d}x_N \, e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x}} = \sqrt{\frac{(2\pi)^N}{\det \mathbf{A}}} e^{\frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{b}}$$

we can calculate moments using the same trick. For $\langle x_i x_j \rangle$ we have

$$\begin{aligned}
\langle x_i x_j \rangle &= \frac{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x_1 \dots \mathrm{d}x_N \, x_i x_j e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}}{\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x_1 \dots \mathrm{d}x_N \, e^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}} \\
&= \sqrt{\frac{\det \mathbf{A}}{(2\pi)^N}} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K} \bigg|_{\mathbf{b}=0} \\
&= \frac{\partial}{\partial b_j} \Big(b_i \big(\mathbf{A}^{-1}\big)_{ik} e^{\frac{1}{2}\mathbf{b}^{\mathrm{T}}\mathbb{A}^{-1}\mathbf{b}} \Big)_{\mathbf{b}=0} \\
\boxed{\langle x_i x_j \rangle = \big(\mathbf{A}^{-1}\big)_{ij}}
\end{aligned}$$

In analogy with the one dimensional case, we see that we can understand these terms diagrammatically by associating factors of $\mathbf{A}^{-1}$ to lines connecting points labeled with factors of x_i. Using this, we can write down the expression for $\langle x_i x_j x_k x_\ell \rangle$ as

$$\boxed{\langle x_i x_j x_k x_\ell \rangle = \big(\mathbf{A}^{-1}\big)_{ij}\big(\mathbf{A}^{-1}\big)_{k\ell} + \big(\mathbf{A}^{-1}\big)_{ik}\big(\mathbf{A}^{-1}\big)_{j\ell} + \big(\mathbf{A}^{-1}\big)_{i\ell}\big(\mathbf{A}^{-1}\big)_{jk}}$$

which can be confirmed by doing the necessary derivatives. For the general case $\langle x_i x_j \dots x_k \rangle$, we have

$$\boxed{\langle x_i x_j \dots x_k x_\ell \rangle = \sum_{\text{Wick}} \big(\mathbf{A}^{-1}\big)_{ab} \cdots \big(\mathbf{A}^{-1}\big)_{cd}}$$

where the indices $\{a, b, \dots c, d\}$ represent a permutation of $\{i, j, \dots, k, \ell\}$, and a "Wick sum" is defined as a sum over all permutations of the indices.

23.3

Consider the Lagrangian

$$L = \frac{1}{2} m \dot{x}^2(t) - \frac{1}{2} m \omega^2 x^2(t) + f(t) x(t)$$

where

$$f(t) = \begin{cases} f_0 & 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

The amplitude $\mathcal{A}$ for a particle to be in the ground state at $t = 0$ and $t = T$ is given by

$$\mathcal{A} = \int_{x(0)=0}^{x(T)=0} \mathcal{D}[x] \exp\left(i \int_{-\infty}^{\infty} \mathrm{d}t \, L \right)$$

Assuming the ground state corresponds to $x = 0$, we can do an integration by parts on the first term in the Lagrangian to write

$$L = \frac{1}{2} x(t) \left[m \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega^2 \right) \right] x(t) + f(t) x(t)$$

Defining

$$\mathcal{C} = m\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega^2\right)$$

we have

$$\mathcal{A} = \int \mathcal{D}[x] \exp\left(i\int_{-\infty}^{\infty} \mathrm{d}t \left(\frac{1}{2}x(t)\mathcal{C}x(t) + f(t)x(t)\right)\right)$$

which is just an inhomogeneous Gaussian integral. Therefore, the solution is given by

$$\mathcal{A} = \frac{B}{\sqrt{\det \mathcal{C}}} \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{d}t'\, f(t)G(t,t')f(t')\right)$$

where $G(t,t)$ is the Green's function to $\mathcal{C}$. As was done in the chapter, we can rewrite this to eliminate B using the free particle propagator as

$$\mathcal{A} = \sqrt{\frac{-im\omega}{2\pi \sin \omega T}} \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{d}t'\, f(t)G(t,t')f(t')\right)$$

The Green's function can be written in frequency space as

$$\tilde{G}(\nu) = \frac{-1}{m}\frac{1}{\nu^2 - \omega^2 + i\epsilon}$$

which we can then inverse Fourier transform to write

$$G(t,t') = \frac{-1}{m}\int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi}\frac{e^{-i\nu(t-t')}}{\nu^2 - \omega^2 + i\epsilon}$$

Our familiarity with the propagator in this form allows us to quickly do the integral by closing the contour either above or below and obtain

$$G(t,t') = \frac{1}{2m\omega}\left(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{i\omega(t-t')}\right)$$

Since the prefactor in the amplitude is there where we have a source or not, we'll focus on just the exponential piece and write

$$\boxed{\begin{aligned} \mathcal{A} &\sim \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{d}t'\, f(t)G(t,t')f(t')\right) \\ G(t,t') &= \frac{1}{2m\omega}\left(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{i\omega(t-t')}\right) \end{aligned}}$$

With the Green's function in this form, the integral in the exponential is straightforward

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{d}t'\, f(t)G(t,t')f(t') &= \frac{f_0^2}{2m\omega}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t' \left(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{i\omega(t-t')}\right) \\
&= \frac{f_0^2}{m\omega}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t'\, e^{-i\omega(t-t')} \\
&= \frac{f_0^2}{m\omega}\int_0^T \mathrm{d}t\, \frac{1}{i\omega}\left(1 - e^{-i\omega t}\right) \\
&= -\frac{if_0^2}{m\omega^2}\left(T - \frac{1}{i\omega}\left(1 - e^{-i\omega T}\right)\right) \\
&= -\frac{if_0^2}{m\omega^2}\left[T + \frac{i}{\omega}\left(\cos\frac{\omega T}{2} - i\sin\frac{\omega T}{2}\right)2i\sin\frac{\omega T}{2}\right] \\
\int_{-\infty}^{\infty} \mathrm{d}t\,\mathrm{d}t'\, f(t)G(t,t')f(t') &= -\frac{if_0^2}{m\omega^2}\left[T - \frac{\sin\omega T}{\omega} + i\frac{2}{\omega}\sin^2\frac{\omega T}{2}\right]
\end{aligned}$$

and therefore we have

$$\boxed{\mathcal{A} = \exp\left[\frac{if_0^2}{2m\omega^2}\left(T - \frac{\sin\omega T}{\omega} + i\frac{2}{\omega}\sin^2\frac{\omega T}{2}\right)\right]}$$

The probability amplitude follows as

$$\boxed{|\mathcal{A}|^2 = \exp\left(-\frac{2f_0^2}{m\omega^3}\sin^2\frac{\omega T}{2}\right)}$$

which is the same result obtained in Exercise 22.1 by expanding the S-operator. The imaginary part of the amplitude corresponds to the complex phase the particle acquires from $t = 0$ to $t = T$. Physically, this is the term which would lead to interference if we were considering a theory in which multiple particles were emitted and absorbed by the source terms.