Chapter 23

Path integrals: I said to him, ‘You’re crazy’

23.1

For the Lagrangian

\[ L = \frac{1}{2} x \dot{A} x + bx \]

assuming \( x \) is an implicit function of \( t \), the equations of motion yield

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0
\]

\[
\dot{A} x + b = 0
\]

\[
x = -\dot{A}^{-1} b
\]

Plugging this in, the Lagrangian evaluated along the equation of motion is given by

\[
L_{\text{EOM}} = \frac{1}{2} \left(-\dot{A}^{-1} b\right) \dot{A} \left(-\dot{A}^{-1} b\right) + b \left(-\dot{A}^{-1} b\right)
\]

\[
= \frac{1}{2} b \dot{A}^{-1} b - b \dot{A}^{-1} b
\]

\[
L_{\text{EOM}} = -b \frac{1}{2} \dot{A}^{-1} b
\]

Since this is the expression which appears in the evaluation of the Gaussian integral

\[
\int \mathcal{D}[x] \exp \left( \frac{i}{2} x \dot{A} x + bx \right) = \frac{B}{\sqrt{\det \dot{A}}} \exp \left( -ib \frac{1}{2} \dot{A}^{-1} b \right)
\]

we see that the exponential is just the classical action evaluated along the classical equation of motion

\[
\int \mathcal{D}[x] \exp \left( \frac{i}{2} x \dot{A} x + bx \right) = \frac{B}{\sqrt{\det \dot{A}}} \exp \left( iS_{\text{EOM}} \right)
\]

23.2

The second moment of the Gaussian distribution

\[
\mu_2 = \int_{-\infty}^{\infty} dx x^2 e^{-\frac{1}{2}a x^2}
\]

can be solved for using the following trick. Since the zeroth moment of the inhomogeneous Gaussian distribution is given by

\[
\mu_{0,b} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}a x^2 + bx} = \sqrt{\frac{2\pi}{a}} e^{b^2/2a}
\]
we can write the homogeneous second moment as
\[ \mu_2 = \frac{\partial^2}{\partial b^2} \left( \mu_{0,b} \right) \bigg|_{b=0} \]

Using the result of the zeroth moment, we find
\[ \mu_2 = \sqrt{\frac{2\pi}{a}} \]

Denoting the \( n \)-th normalized moment as \( \langle x^n \rangle \)
\[ \langle x^n \rangle = \frac{\int_{-\infty}^{\infty} dx \, x^n e^{-\frac{1}{2} ax^2}}{\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} ax^2}} \]
we can work out the first two non-zero moments as follows
\[ \langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} dx \, x^2 e^{-\frac{1}{2} ax^2}}{\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} ax^2}} = \sqrt{\frac{a}{2\pi}} \left( \frac{\partial^2}{\partial b^2} \left( \mu_{0,b} \right) \right) \bigg|_{b=0} \]
\[ \langle x^2 \rangle = \frac{1}{a} \]

\[ \langle x^4 \rangle = \frac{\int_{-\infty}^{\infty} dx \, x^4 e^{-\frac{1}{2} ax^2}}{\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2} ax^2}} = \sqrt{\frac{a^2}{2\pi}} \left( \frac{\partial^4}{\partial b^4} \left( \mu_{0,b} \right) \right) \bigg|_{b=0} \]
\[ \langle x^4 \rangle = \frac{3}{a^2} \]

In order to really see the pattern, we can go one step further and find that \( \langle x^6 \rangle = \frac{15}{a^3} \), at which point we can extrapolate and write
\[ \langle x^n \rangle = \begin{cases} 0 & n \text{ odd} \\ \frac{(n-1)!!}{a^{n/2}} & n \text{ even} \end{cases} \]

which can be checked and is indeed correct. This can be understood diagrammatically by associating each factor of \( 1/a \) with a line which connects two factors of \( x \). For \( \langle x^2 \rangle \), there are only two factors of \( x \), and thus only one way to connect them

\[ \langle x^2 \rangle = x \rightarrow x \]

As for \( \langle x^4 \rangle \), there are three ways to combine four factors of \( x \) into pairs

\[ \langle x^4 \rangle = \begin{array}{c} x \rightarrow x \rightarrow x \rightarrow x \\ + x \rightarrow x \leftrightarrow x \rightarrow x \\ + x \rightarrow x \rightarrow x \rightarrow x \end{array} \]
Therefore, for \( n \) even we have

\[
\langle x^n \rangle = \sum_{\text{All graphs}} (\text{n points connected into pairs})
\]

Upgrading to the \( N \)-dimensional Gaussian integral

\[
\mathcal{K} = \int_{-\infty}^{\infty} dx_1 \ldots dx_N e^{-\frac{1}{2} x^T A x + b^T x} = \sqrt{\frac{(2\pi)^N}{\det A}} e^{\frac{1}{2} b^T A^{-1} b}
\]

we can calculate moments using the same trick. For \( \langle x_i x_j \rangle \) we have

\[
\langle x_i x_j \rangle = \int_{-\infty}^{\infty} dx_1 \ldots dx_N x_i x_j e^{-\frac{1}{2} x^T A x} = \int_{-\infty}^{\infty} dx_1 \ldots dx_N e^{-\frac{1}{2} x^T A x} = \sqrt{\frac{(2\pi)^N}{\det A}} \left( \frac{\partial^2}{\partial b_j \partial b_i} \mathcal{K} \right)_{b=0} = \frac{\partial^2}{\partial b_j} \left( b_i (A^{-1})_{ik} e^{\frac{1}{2} b^T A^{-1} b} \right)_{b=0}
\]

\[
\langle x_i x_j \rangle = (A^{-1})_{ij}
\]

In analogy with the one dimensional case, we see that we can understand these terms diagrammatically by associating factors of \( A^{-1} \) to lines connecting points labeled with factors of \( x_i \). Using this, we can write down the expression for \( \langle x_i x_j x_k x_\ell \rangle \) as

\[
\langle x_i x_j x_k x_\ell \rangle = (A^{-1})_{ij} (A^{-1})_{kl} + (A^{-1})_{ik} (A^{-1})_{j\ell} + (A^{-1})_{il} (A^{-1})_{jk}
\]

which can be confirmed by doing the necessary derivatives. For the general case \( \langle x_i x_j \ldots x_k \rangle \), we have

\[
\langle x_i x_j \ldots x_k \rangle = \sum_{\text{Wick}} (A^{-1})_{ab} \ldots (A^{-1})_{cd}
\]

where the indices \( \{a, b, \ldots c, d\} \) represent a permutation of \( \{i, j, \ldots, k, \ell\} \), and a “Wick sum” is defined as a sum over all permutations of the indices.

### 23.3

Consider the Lagrangian

\[
L = \frac{1}{2} m \dot{x}^2(t) - \frac{1}{2} m \omega^2 x^2(t) + f(t)x(t)
\]

where

\[
f(t) = \begin{cases} 
  f_0 & 0 \leq t \leq T \\
  0 & \text{else}
\end{cases}
\]

The amplitude \( A \) for a particle to be in the ground state at \( t = 0 \) and \( t = T \) is given by

\[
A = \int_{x(0)=0}^{x(T)=0} D[x] \exp \left( i \int_{-\infty}^{\infty} dt L \right)
\]

Assuming the ground state corresponds to \( x = 0 \), we can do an integration by parts on the first term in the Lagrangian to write

\[
L = \frac{1}{2} x(t) \left[ m \left( -\frac{d^2}{dt^2} - \omega^2 \right) x(t) + f(t)x(t) \right]
\]
Defining

\[ C = m \left( -\frac{d^2}{dt^2} - \omega^2 \right) \]

we have

\[ A = \int D[x] \exp \left( i \int_{-\infty}^{\infty} dt \left( \frac{1}{2} x(t) C x(t) + f(t) x(t) \right) \right) \]

which is just an inhomogeneous Gaussian integral. Therefore, the solution is given by

\[ A = \frac{B}{\sqrt{\det C}} \exp \left( -\frac{1}{2} \int_{-\infty}^{\infty} dt \, dt' \, f(t) G(t, t') f(t') \right) \]

where \( G(t, t) \) is the Green’s function to \( C \). As was done in the chapter, we can rewrite this to eliminate \( B \) using the free particle propagator as

\[ A = \sqrt{-\frac{im\omega}{2\pi \sin \omega T}} \exp \left( -\frac{1}{2} \int_{-\infty}^{\infty} dt \, dt' \, f(t) G(t, t') f(t') \right) \]

The Green’s function can be written in frequency space as

\[ \tilde{G}(\nu) = \frac{-1}{m} \frac{1}{\nu^2 - \omega^2 + i\epsilon} \]

which we can then inverse Fourier transform to write

\[ G(t, t') = \frac{-1}{m} \int_{-\infty}^{\infty} d\nu \, \frac{e^{-i\nu(t-t')}}{2\pi \nu^2 - \omega^2 + i\epsilon} \]

Our familiarity with the propagator in this form allows us to quickly do the integral by closing the contour either above or below and obtain

\[ G(t, t') = \frac{1}{2m\omega} \left( \theta(t-t') e^{-i\omega(t-t')} + \theta(t'-t) e^{i\omega(t-t')} \right) \]

Since the prefactor in the amplitude is there where we have a source or not, we’ll focus on just the exponential piece and write

\[
\begin{align*}
A &\sim \exp \left( -\frac{1}{2} \int_{-\infty}^{\infty} dt \, dt' \, f(t) G(t, t') f(t') \right) \\
G(t, t') &= \frac{1}{2m\omega} \left( \theta(t-t') e^{-i\omega(t-t')} + \theta(t'-t) e^{i\omega(t-t')} \right)
\end{align*}
\]

With the Green’s function in this form, the integral in the exponential is straightforward

\[
\begin{align*}
\int_{-\infty}^{\infty} dt \, dt' \, f(t) G(t, t') f(t') &= \int_{-\infty}^{T} dt \int_{0}^{T} dt' \, \left( \theta(t-t') e^{-i\omega(t-t')} + \theta(t'-t) e^{i\omega(t-t')} \right) \\
&= \frac{f_0^2}{2m\omega} \int_{0}^{T} dt \int_{0}^{T} dt' \, e^{-i\omega(t-t')} \\
&= \frac{f_0^2}{m\omega} \int_{0}^{T} dt \, \frac{1}{i\omega} \left( 1 - e^{-i\omega t} \right) \\
&= -\frac{if_0^2}{m\omega^2} \left( T - \frac{1}{i\omega} \left( 1 - e^{-i\omega T} \right) \right) \\
&= -\frac{if_0^2}{m\omega^2} \left[ T + \frac{i}{\omega} \left( \frac{\omega T}{2} - i \frac{\sin \frac{\omega T}{2}}{2} \right) 2i \sin \frac{\omega T}{2} \right] \\
\int_{-\infty}^{\infty} dt \, dt' \, f(t) G(t, t') f(t') &= -\frac{if_0^2}{m\omega^2} \left[ T - \frac{\sin \omega T}{\omega} + i \frac{2}{\omega} \sin^2 \frac{\omega T}{2} \right]
\end{align*}
\]

and therefore we have

\[ A = \exp \left[ \frac{if_0^2}{2m\omega^2} \left( T - \frac{\sin \omega T}{\omega} + i \frac{2}{\omega} \sin^2 \frac{\omega T}{2} \right) \right] \]
The probability amplitude follows as

\[ |A|^2 = \exp \left( -\frac{2f_0^2}{m\omega^3} \sin^2 \frac{\omega T}{2} \right) \]

which is the same result obtained in Exercise 22.1 by expanding the S-operator. The imaginary part of the amplitude corresponds to the complex phase the particle acquires from \( t = 0 \) to \( t = T \). Physically, this is the term which would lead to interference if we were considering a theory in which multiple particles were emitted and absorbed by the source terms.