Chapter 22

The generating functional for fields

22.1

Consider the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\omega^2 x^2(t) + f(t)x(t)$$

and corresponding Hamiltonian

$$H = \underbrace{\frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}m\omega^2 x^2(t)}_{H_0} + \underbrace{\Big(-f(t)x(t)\Big)}_{H_I}$$

where

$$f(t) = \begin{cases} f_0 & 0 < t < T \\ 0 & \text{else} \end{cases}$$

The generating functional (ignoring normalization) is defined as

$$Z[f] = \langle\Omega|U(\infty,-\infty)|\Omega\rangle$$

where $|\Omega\rangle$ is the interacting ground state, and $U(t,t')$ is the time evolution operator in the Heisenberg picture. Using the definition of the interaction picture, we have

$$Z[f] = \lim_{t\to\infty} \langle\Omega|e^{-iH_0 t}U_I(t,-t)e^{iH_0 t}|\Omega\rangle$$

In the limit $t \to \pm\infty$, the physical ground state for this theory coincides with the bare ground state up to a phase which is taken care of by the normalization of the generating functional. Since the bare ground state has energy set to zero, $H_0\,|0\rangle = 0$, we have

$$\begin{aligned} Z[f] &= \lim_{t\to\infty} \langle 0|e^{-iH_0 t}U_I(t,-t)e^{iH_0 t}|0\rangle \\ &= \lim_{t\to\infty} \langle 0|U_I(t,-t)|0\rangle \\ &\boxed{Z[f] = \langle 0|S|0\rangle} \end{aligned}$$

Expanding the S operator, we can write

$$\langle 0|S|0\rangle = \langle 0|T\exp\Big(-i\int_{-\infty}^{\infty} \mathrm{d}t\, H_I\Big)|0\rangle$$

$$\langle 0|S|0\rangle = \langle 0|0\rangle + i\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\,\langle 0|x(t)|0\rangle + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}t'\, f(t)\,\langle 0|T\Big[x(t)x(t')\Big]|0\rangle\, f(t') + \dots$$

Recalling the expansion of the position operator in the interaction picture

$$x(t) = \frac{1}{\sqrt{2m\omega}}\big(ae^{-i\omega t} + a^\dagger e^{i\omega t}\big)$$

we can see that vacuum expectation values with an odd number of position operators will vanish, leaving only the even powers. Using Wick's theorem we can then write

$$
\begin{aligned}
Z[f] &= 1 + \frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|\overbracket{x(t)x(t')}|0\rangle\, f(t') \\
&\quad + \frac{(-i)^2}{4!}\int \mathrm{d}t_1\,\mathrm{d}t_1'\,\mathrm{d}t_2\,\mathrm{d}t_2'\; f(t_1)f(t_1')\,3\,\langle 0|\overbracket{x(t_1)x(t_1')}\overbracket{x(t_2)x(t_2')}|0\rangle\, f(t_2)f(t_2') + \dots \\
&= 1 + \frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|\overbracket{x(t)x(t')}|0\rangle\, f(t') \\
&\quad + \frac{1}{2!}\left(\frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|\overbracket{x(t)x(t')}|0\rangle\, f(t')\right)^2 + \dots \\
&= \exp\left(\frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|\overbracket{x(t)x(t')}|0\rangle\, f(t')\right) \\
&= \exp\left(\frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|T\Big[x(t)x(t')\Big]|0\rangle\, f(t')\right)
\end{aligned}
$$

$$
\boxed{Z[f] = \exp\Big(\text{Dumbbell}\Big)}
$$

where

$$
\text{Dumbbell} = \frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|T\Big[x(t)x(t')\Big]|0\rangle\, f(t')
$$

This agrees with the diagrammatic method

$$
Z[f] = \exp\Big(\sum \text{connected diagrams}\Big)
$$

since the Dumbbell diagram shown below is the only connected diagram for this theory

In frequency space, we can write the expression for the diagram by recalling the propagator for the quantum oscillator of frequency ω

$$
G(\nu,\omega) = \left(\frac{1}{m}\right)\frac{i}{\nu^2-\omega^2+i\epsilon}
$$

and integrating over ν. As for the source terms, their Fourier transform will appear in the integrand

$$
\tilde{f}(\nu) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)e^{-i\nu t}
$$

Since this is a VEV, if we create an excitation with $f(\nu)$, the other source must annihilate it as $f(-\nu)$. Putting these together, we find

$$
\text{Dumbbell} = \frac{-1}{2m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\tilde{f}(\nu)\frac{i}{\nu^2-\omega^2+i\epsilon}\tilde{f}(-\nu)
$$

Alternatively, we could calculate this directly by using the expansion of the position operator

$$
\begin{aligned}
\text{Dumbbell} &= \frac{(-i)^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\,\langle 0|\Big(\theta(t-t')x(t)x(t') + \theta(t'-t)x(t')x(t)\Big)|0\rangle\, f(t') \\
\text{Dumbbell} &= \frac{-1}{4m\omega}\int \mathrm{d}t\,\mathrm{d}t'\; f(t)\Big(\theta(t-t')e^{-i\omega(t-t')} + \theta(t'-t)e^{i\omega(t-t')}\Big)f(t')
\end{aligned}
$$

Recalling the integral representation of the Heaviside function (with the limit as $\epsilon \to 0$ implied)

$$\theta(x) = i\int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi}\frac{e^{-izx}}{z+i\epsilon}$$

we have

$$\text{Dumbbell} = \frac{-i}{4m\omega}\int \mathrm{d}t\,\mathrm{d}t'\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}f(t)\left(\frac{e^{-i(\nu+\omega)(t-t')}}{\nu+i\epsilon}+\frac{e^{i(\nu+\omega)(t-t')}}{\nu+i\epsilon}\right)f(t')$$

Redefining $\nu \to \nu+\omega$

$$\text{Dumbbell} = \frac{-i}{4m\omega}\int \mathrm{d}t\,\mathrm{d}t'\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}f(t)\left(\frac{e^{-i\nu(t-t')}}{\nu-\omega+i\epsilon}+\frac{e^{i\nu(t-t')}}{\nu-\omega+i\epsilon}\right)f(t')$$

and subsequently $\nu \to -\nu$ in the second integrand

$$\begin{aligned}
\text{Dumbbell} &= \frac{-i}{4m\omega}\int \mathrm{d}t\,\mathrm{d}t'\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}f(t)\left(\frac{1}{\nu-\omega+i\epsilon}-\frac{1}{\nu+\omega-i\epsilon}\right)e^{-i\nu(t-t')}f(t')\\
&= \frac{-i}{2m\omega}\int \mathrm{d}t\,\mathrm{d}t'\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}f(t)\frac{\omega-i\epsilon}{\nu^2-\omega^2+i\epsilon}e^{-i\nu(t-t')}f(t')\\
&= \frac{-1}{2m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\left(\int_{-\infty}^{\infty}\mathrm{d}t\,f(t)e^{-i\nu t}\right)\frac{i}{\nu^2-\omega^2+i\epsilon}\left(\int_{-\infty}^{\infty}\mathrm{d}t'\,f(t')e^{i\nu t'}\right)\\
\text{Dumbbell} &= \frac{-1}{2m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\left(\frac{if_0}{\nu}\left(1-e^{i\nu T}\right)\right)\frac{i}{\nu^2-\omega^2+i\epsilon}\left(\frac{if_0}{-\nu}\left(1-e^{-i\nu T}\right)\right)
\end{aligned}$$

Defining

$$\tilde{f}(\nu) = \frac{if_0}{\nu}\left(1-e^{i\nu T}\right)$$

we have

$$\boxed{\text{Dumbbell} = \frac{-1}{2m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\tilde{f}(\nu)\frac{i}{\nu^2-\omega^2+i\epsilon}\tilde{f}(-\nu)}$$

which agrees exactly with the answer using the Feynman rules for this theory. As for $\left|Z[f]\right|^2$, this physically represents the probability of starting and ending in the ground state. Clearly this will depend on f_0 and T. We can solve for this by using the identity

$$\frac{1}{E+i\epsilon} = \mathcal{P}\frac{1}{E} - i\pi\delta(E)$$

where this equation is understood in the context of multiplying by a test function and integrating. Identifying $E = \nu^2-\omega^2$, we can write

$$\begin{aligned}
\text{Dumbbell} &= \frac{-i}{4\pi m}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{\tilde{f}(\nu)\tilde{f}(-\nu)}{\nu^2-\omega^2} - i\pi\left(\frac{-i}{4\pi m}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\tilde{f}(\nu)\tilde{f}(-\nu)\delta(\nu^2-\omega^2)\right)\\
&= \frac{-i}{4\pi m}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{\tilde{f}(\nu)\tilde{f}(-\nu)}{\nu^2-\omega^2} - \frac{1}{4m}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\tilde{f}(\nu)\tilde{f}(-\nu)\frac{\delta(\nu-\omega)+\delta(\nu+\omega)}{2\omega}\\
&= \frac{-i}{4\pi m}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{\tilde{f}(\nu)\tilde{f}(-\nu)}{\nu^2-\omega^2} - \frac{1}{4m\omega}\tilde{f}(\omega)\tilde{f}(-\omega)\\
\text{Dumbbell} &= \frac{-if_0^2}{\pi m}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{1}{\nu^2}\sin^2\left(\frac{\nu T}{2}\right)\frac{1}{\nu^2-\omega^2} - \frac{f_0^2}{m\omega^3}\sin^2\left(\frac{\omega T}{2}\right)
\end{aligned}$$

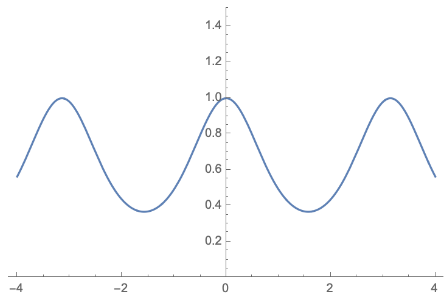
With this, $Z[f]$ can be written as

$$Z[f] = \exp\left(\frac{-if_0^2}{\pi m}\mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}\nu\,\frac{1}{\nu^2}\sin^2\left(\frac{\nu T}{2}\right)\frac{1}{\nu^2-\omega^2}\right)\exp\left(-\frac{f_0^2}{m\omega^3}\sin^2\left(\frac{\omega T}{2}\right)\right)$$

Since the principal value of this integral is a real number, it's contribution to $|Z[f]|^2$ will vanish, leaving the result

$$\boxed{|Z[f]|^2 = \exp\left(-\frac{2f_0^2}{m\omega^3}\sin^2\left(\frac{\omega T}{2}\right)\right)}$$

Focusing on the T dependence, a plot of $\exp\left(-\sin^2 x\right)$ can be seen below, which exhibits the same functional behavior of our solution



The probability to start and end in the ground state is always non-zero, and is equal to unity at $T = 0$, which makes sense. Interestingly though, the probability is periodic, and therefore for certain durations of the interaction, the probability of starting and ending in the ground state is again exactly unity. This makes sense classically if we imagine putting our system in the minimum of a harmonic oscillator potential with some initial momentum. It will oscillate back and forth in the potential and therefore we will find it exactly at the minimum whenever the time elapsed equals a multiple of the period of oscillation.

22.2

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi(x))^2 - \frac{m^2}{2}\varphi(x)^2 + gJ(x)\varphi(x)$$

we have the dumbbell diagram shown below

at $\mathcal{O}(g^2)$ in the VEV of the S operator

$$\begin{aligned}
\text{Dumbbell} &= \frac{(-ig)^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\,\langle 0|\varphi(x)\varphi(y)|0\rangle\,J(y) \\
&= \frac{(-ig)^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,J(x)\Delta(x-y)J(y)
\end{aligned}$$

$$\boxed{\text{Dumbbell} = \frac{(-ig)^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\int\frac{\mathrm{d}^4p}{(2\pi)^4}J(x)\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}J(y)}$$

For a source given by two static sources

$$J(x) = \delta^{(3)}(\mathbf{x}-\mathbf{x}_1) + \delta^{(3)}(\mathbf{x}-\mathbf{x}_2)$$

we can write (neglecting the nonphysical self-interaction terms)

$$\begin{aligned}
\text{Dumbbell}_{12} &= \frac{(-ig)^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \\
&\quad\times\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\delta^{(3)}(\mathbf{y}-\mathbf{x}_2)+\delta^{(3)}(\mathbf{y}-\mathbf{x}_1)\delta^{(3)}(\mathbf{x}-\mathbf{x}_2)\Big) \\
&= \frac{(-ig)^2}{2}\int \mathrm{d}x^0\,\mathrm{d}y^0 \int \frac{\mathrm{d}p^0}{2\pi}e^{-ip^0(x^0-y^0)}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i}{p^2-m^2+i\epsilon} \\
&\quad\times\Big(e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}+e^{i\mathbf{p}\cdot(\mathbf{x}_2-\mathbf{x}_1)}\Big)
\end{aligned}$$

$$\boxed{\text{Dumbbell}_{12} = -ig^2\int \mathrm{d}x^0\,\mathrm{d}y^0\int\frac{\mathrm{d}p^0}{2\pi}e^{-ip^0(x^0-y^0)}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{p^2-m^2+i\epsilon}}$$

Furthermore, we can perform the y^0 integral which sets $p^0=0$

$$\begin{aligned}
\text{Dumbbell}_{12} &= -ig^2\int \mathrm{d}x^0\int\frac{\mathrm{d}p^0}{2\pi}\int \mathrm{d}y^0\,e^{-ip^0(x^0-y^0)}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{p^2-m^2+i\epsilon} \\
&= -ig^2\int \mathrm{d}x^0\int \mathrm{d}p^0\,\delta(p^0)e^{-ip^0x^0}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{p^2-m^2+i\epsilon}
\end{aligned}$$

$$\boxed{\text{Dumbbell}_{12} = ig^2\int \mathrm{d}x^0\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{p}^2+m^2}}$$

where we can drop the $+i\epsilon$ as the denominator is always positive. For this particular source, the interaction Hamiltonian becomes

$$\begin{aligned}
H_I &= -g\int \mathrm{d}^3x\,\varphi(t,\mathbf{x})J(x) \\
&= -g\varphi(t,\mathbf{x}_1)-g\varphi(t,\mathbf{x}_2) \\
H_I &= -ge^{iH_0t}\Big(\varphi(\mathbf{x}_1)+\varphi(\mathbf{x}_2)\Big)e^{-iH_0t}
\end{aligned}$$

where we've used the definition of interaction picture operators to remove their time-dependence. If we then consider the VEV of the S-matrix, since $H_0|0\rangle=0$, the time dependence of H_I vanishes and we have

$$\langle 0|S|0\rangle = \langle 0|\exp\left(-i\int \mathrm{d}x^0\,H_I\right)|0\rangle = \exp\left(-iE\int \mathrm{d}x^0\right)$$

where E is the energy of the interaction in the ground state. Comparing this with our derived expression

$$\langle 0|S|0\rangle = \exp\Big(\text{Dumbbell}\Big) = \exp\left(ig^2\int \mathrm{d}x^0\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{p}^2+m^2}\right)$$

we can read off the interaction energy between the two point-like sources is given by

$$\boxed{E = -g^2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{p}^2+m^2}}$$

Defining $r=|\mathbf{x}_1-\mathbf{x}_2|$, this integral can be performed in spherical coordinates via the residue theorem and yields

$$E = \frac{-g^2}{4\pi r}e^{-mr}$$

which is the well-known Yukawa interaction potential, with characteristic length scale m^{-1}. Since this energy is negative, we arrive at the conclusion that a spin zero "particle" of mass m exchanged between two point-like, static sources produces an attractive force. Since protons and neutrons experience the strong force via the exchange of a pion (a spin zero particle), this is a rough argument for why the strong force is attractive at the level of nucleons.