Chapter 21

Statistical physics: a crash course

21.1

For the general partition function

$$Z(J) = \mathrm{Tr}\left[e^{-\beta H(A)+JA}\right]$$

we can write

$$\langle A\rangle_t = \mathrm{Tr}\left[A\rho\right] = \frac{1}{Z(0)}\,\mathrm{Tr}\left[Ae^{-\beta H(A)}\right] = \frac{1}{Z(0)}\frac{\partial}{\partial J}Z(J)\bigg|_{J=0}$$

For the quantum oscillator Hamiltonian

$$H = \omega a^\dagger a = \omega N$$

the partition function is given by

$$\begin{aligned} Z(J) &= \sum_n \langle n|e^{-\beta\omega N+JN}|n\rangle \\ &= \sum_n \left(e^{-\beta\omega+J}\right)^n \\ Z(J) &= \frac{1}{1-e^{-\beta\omega+J}} \end{aligned}$$

With this, the thermal expectation value of the number operator is given by

$$\begin{aligned} \langle N\rangle_t &= \frac{1}{Z(0)}\frac{\partial}{\partial J}Z(J)\bigg|_{J=0} \\ &= \left(1-e^{-\beta\omega}\right)\frac{e^{-\beta\omega}}{\left(1-e^{-\beta\omega}\right)^2} \end{aligned}$$

$$\boxed{\langle N\rangle_t = \frac{1}{e^{\beta\omega}-1}}$$

21.2

The forced quantum oscillator has a Lagrangian

$$L = \frac{1}{2}m\dot{x}^2(t) - \frac{1}{2}m\omega^2x^2(t) + f(t)x(t)$$

Treating the driving term as an interaction in the Hamiltonian framework

$$H_0(t) = \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}m\omega^2x^2(t), \quad H'(t) = -f(t)x(t)$$

we can work in the interaction picture and write

$$H_I'(t) = -e^{iH_0t}f(t)x(t)e^{-iH_0t} = -f_I(t)x_I(t)$$

Now, consider a general state $|\psi_I(t)\rangle$. Assuming $f_I(t \to -\infty) \to 0$, then $|\psi_I(t \to -\infty)\rangle$ can be built up from the the ground state of the quantum oscillator, $|0\rangle$. For the purposes of this problem, we'll assume that as $t \to -\infty$, our state is the ground state

$$|\psi(t \to -\infty)\rangle = |0\rangle$$

With this, we can use Dyson's expansion and work to $\mathcal{O}(f_I(t))$ to write

$$\begin{aligned}
|\psi_I(t)\rangle &= U(t, -\infty)\,|0\rangle \\
&= T\left[\exp\left\{-i\int_{-\infty}^{t} \mathrm{d}t'\, H_I'(t')\right\}\right]|0\rangle \\
&= \left[1 - i\int_{-\infty}^{t} \mathrm{d}t'\, H_I'(t') + \mathcal{O}\big(f_I^2(t)\big)\right]|0\rangle \\
\Aboxed{|\psi_I(t)\rangle &= |0\rangle + i\int_{-\infty}^{t} \mathrm{d}t'\, f_I(t')x_I(t')\,|0\rangle}
\end{aligned}$$

With this, we can calculate the expectation value of the position operator in this state to $\mathcal{O}(f_I(t))$

$$\begin{aligned}
\langle\psi_I(t)|x_I(t)|\psi_I(t)\rangle &= \bigg(\langle 0| - i\int_{-\infty}^{t} \mathrm{d}t'\, f_I(t')x_I(t')\,\langle 0|\bigg)x_I(t)\bigg(|0\rangle + i\int_{-\infty}^{t} \mathrm{d}t'\, f_I(t')x_I(t')\,|0\rangle\bigg) \\
&= \langle 0|x_I(t)|0\rangle + i\int_{-\infty}^{t} \mathrm{d}t'\, f_I(t')\,\langle 0|[x_I(t), x_I(t')]|0\rangle + \mathcal{O}(f_I^2(t)) \\
\langle\psi_I(t)|x_I(t)|\psi_I(t)\rangle &= i\int_{-\infty}^{\infty} \mathrm{d}t'\, \theta(t-t')f_I(t')\,\langle 0|[x_I(t), x_I(t')]|0\rangle
\end{aligned}$$

Comparing this with the definition of the response function

$$\langle\psi_I(t)|x_I(t)|\psi_I(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}t'\, \chi(t-t')f(t')$$

we can read off

$$\boxed{\chi(t-t') = i\theta(t-t')\,\langle 0|[x_I(t), x_I(t')]|0\rangle}$$

Using the decomposition

$$x_I(t) = \frac{1}{\sqrt{2m\omega}}\Big(ae^{-i\omega t} + a^\dagger e^{i\omega t}\Big)$$

we can evaluate the commutator as

$$\begin{aligned}
[x_I(t), x_I(t')] &= \frac{1}{2m\omega}\Big[\Big(ae^{-i\omega t} + a^\dagger e^{i\omega t}\Big), \Big(ae^{-i\omega t'} + a^\dagger e^{i\omega t'}\Big)\Big] \\
&= \frac{1}{2m\omega}\Big(e^{-i\omega(t-t')} - e^{i\omega(t-t')}\Big)[a, a^\dagger] \\
[x_I(t), x_I(t')] &= \frac{-i}{m\omega}\sin\left(\omega(t-t')\right)
\end{aligned}$$

which yields a response function

$$\boxed{\chi(t-t') = \frac{1}{m\omega}\sin\left(\omega(t-t')\right)\theta(t-t')}$$

At non-zero T, we can follow a similar procedure as above to and relate the density matrix $\rho(t)$ to the equilibrium density matrix ρ_t in an expansion up to $\mathcal{O}(f_I(t))$. This yields an expression for the response function

$$\chi(t-t') = i\theta(t-t')\,\langle [x_I(t), x_I(t')]\,\rangle_t$$

Evaluating the expectation value yields

$$\begin{aligned}
\langle [x_I(t), x_I(t')] \rangle_t &= \frac{1}{Z} \operatorname{Tr}\left(e^{-\beta\omega N}[x_I(t), x_I(t')]\right) \\
&= \left(1 - e^{-\beta\omega}\right) \sum_n e^{-\beta\omega n} \langle n|[x_I(t), x_I(t')]|n\rangle \\
&= \frac{-i}{m\omega} \sin(\omega(t-t'))\left(1 - e^{-\beta\omega}\right) \sum_n e^{-\beta\omega n} \langle n|n\rangle \\
\langle [x_I(t), x_I(t')] \rangle_t &= \frac{-i}{m\omega} \sin(\omega(t-t'))
\end{aligned}$$

The fact that the results are the same is not too surprising, as the result at zero temperature did not depend on n. Therefore the response function is simply

$$\boxed{\chi(t-t') = \frac{1}{m\omega} \sin\left(\omega(t-t')\right)\theta(t-t')}$$

We can compare this result to that of the correlation function defined by

$$S = \frac{1}{2} \langle \{x_I(t'), x_I(t)\} \rangle$$

Taking the expectation value first at zero temperature, we have

$$\begin{aligned}
S &= \frac{1}{4m\omega} \langle \left\{ \left(ae^{-i\omega t'} + a^\dagger e^{i\omega t'}\right), \left(ae^{-i\omega t} + a^\dagger e^{i\omega t}\right) \right\} \rangle \\
&= \frac{1}{4m\omega} \langle \left(aa^\dagger + a^\dagger a\right) e^{i\omega(t-t')} + \left(aa^\dagger + a^\dagger a\right) e^{-i\omega(t-t')} \rangle \\
&= \frac{1}{2m\omega} \langle (1+2N)\cos(\omega(t-t')) \rangle \\
S &= \frac{1}{m\omega} \cos(\omega(t-t')) \left(\langle N \rangle + \frac{1}{2}\right)
\end{aligned}$$

For the ground state, we have

$$\boxed{S = \frac{1}{2m\omega} \cos(\omega(t-t'))}$$

At non-zero temperature, we can simply evaluate the expectation value in the above equation in thermal equilibrium

$$S = \frac{1}{m\omega} \cos(\omega(t-t')) \left(\langle N \rangle_t + \frac{1}{2}\right)$$

Using the result of the previous problem, we obtain

$$\boxed{S = \frac{1}{m\omega} \cos(\omega(t-t')) \left(\frac{1}{e^{\beta\omega} - 1} + \frac{1}{2}\right)}$$

From this, we learn that for the quantum oscillator, the two-time position response function does not distinguish thermal and quantum fluctuations, while the two-time position correlation function does and reduces to the case of purely quantum fluctuations in the limit $\beta \to \infty$.

21.3

Consider the diffusion equation

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) n(\mathbf{x}, t) = 0$$

The Green's function for this equation satisfies

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) G(\mathbf{x} - \mathbf{y}, t) = \delta(\mathbf{x} - \mathbf{y})\delta(t)$$

which for $t > 0$ is equivalent to

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) G(\mathbf{x}-\mathbf{y},t) = 0$$

Taking a Laplace transform in time with a purely imaginary Laplace variable, $s = -i\omega$, takes us to the frequency domain, yielding

$$-i\omega\tilde{G}(\mathbf{x}-\mathbf{y},\omega) - G(\mathbf{x}-\mathbf{y},0) - D\nabla^2\tilde{G}(\mathbf{x}-\mathbf{y},\omega) = 0$$

For the boundary condition, $G(\mathbf{x}-\mathbf{y},0) = \delta(\mathbf{x}-\mathbf{y})$, we have

$$-i\omega\tilde{G}(\mathbf{x}-\mathbf{y},\omega) - D\nabla^2\tilde{G}(\mathbf{x}-\mathbf{y},\omega) = \delta(\mathbf{x}-\mathbf{y})$$

which we can then solve via a Fourier transform in space

$$-i\omega\tilde{G}(\mathbf{q},\omega) + D\mathbf{q}^2\tilde{G}(\mathbf{q},\omega) = 1$$

$$\boxed{\tilde{G}(\mathbf{q},\omega) = \frac{1}{-i\omega + D\mathbf{q}^2}}$$