

Chapter 20

Scattering Theory

20.1

For free fields, all Wick contractions between fields and between a field and a creation/annihilation operator will be c -numbers. Therefore, we can write them as time-ordered VEVs. For the $\psi^\dagger\psi\phi$ theory, we have the free fields

$$\begin{aligned}\psi(x) &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} (a_{\mathbf{p}} e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x}), \quad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2} \\ \psi^\dagger(x) &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} (a_{\mathbf{p}}^\dagger e^{ip \cdot x} + b_{\mathbf{p}} e^{-ip \cdot x}) \\ \phi(x) &= \int \frac{d^3 q}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_{\mathbf{q}})^{1/2}} (c_{\mathbf{q}} e^{-iq \cdot x} + c_{\mathbf{q}}^\dagger e^{iq \cdot x}), \quad \varepsilon_{\mathbf{q}} = \sqrt{\mathbf{q}^2 + \mu^2}\end{aligned}$$

The only possible, non-zero contractions we can have are $\overline{\psi(x)\psi^\dagger(y)}$, $\overline{\phi(x)\phi(y)}$, $\overline{a_{\mathbf{p}}\psi^\dagger(x)}$, $\overline{b_{\mathbf{p}}\psi(x)}$, $\overline{c_{\mathbf{q}}\phi(x)}$, $\overline{\psi(x)a_{\mathbf{p}}^\dagger}$, $\overline{\psi^\dagger(x)b_{\mathbf{p}}^\dagger}$, and $\overline{\phi(x)c_{\mathbf{q}}^\dagger}$, which evaluate to

$$\begin{aligned}\overline{\psi(x)\psi^\dagger(y)} &= \langle 0 | T[\psi(x)\psi^\dagger(y)] | 0 \rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} (e^{-ip \cdot (x-y)} \theta(x^0 - y^0) + e^{ip \cdot (x-y)} \theta(y^0 - x^0)) \\ &= i \int \frac{dz d^3 p}{(2\pi)^4} \frac{1}{2E_{\mathbf{p}}} \left(\frac{e^{-i(E_{\mathbf{p}}+z)(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{z+i\epsilon} + \frac{e^{i(E_{\mathbf{p}}+z)(x^0-y^0)-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{z+i\epsilon} \right) \\ &= i \int \frac{dz' d^3 p}{(2\pi)^4} \frac{1}{2E_{\mathbf{p}}} \left(\frac{e^{-iz'(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{z'-E_{\mathbf{p}}+i\epsilon} + \frac{e^{iz'(x^0-y^0)-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{z'-E_{\mathbf{p}}+i\epsilon} \right) \\ &= i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{2E_{\mathbf{p}}} e^{-ip \cdot (x-y)} \left(\frac{1}{p^0 - E_{\mathbf{p}} + i\epsilon} - \frac{1}{p^0 + E_{\mathbf{p}} - i\epsilon} \right) \\ \boxed{\overline{\psi(x)\psi^\dagger(y)} = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}}\end{aligned}$$

$$\begin{aligned}\overline{\phi(x)\phi(y)} &= \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle \\ &= \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2\varepsilon_{\mathbf{q}}} (e^{-iq \cdot (x-y)} \theta(x^0 - y^0) + e^{iq \cdot (x-y)} \theta(y^0 - x^0))\end{aligned}$$

$$\boxed{\overline{\phi(x)\phi(y)} = \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - \mu^2 + i\epsilon} e^{-iq \cdot (x-y)}}$$

$$\begin{aligned}\boxed{a_{\mathbf{p}}^{\dagger} \psi^{\dagger}(x)} &= \langle 0 | a_{\mathbf{p}} \psi^{\dagger}(x) | 0 \rangle \\ &= \int \frac{d^3 q}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{q}})^{1/2}} \langle \mathbf{p} | \mathbf{q} \rangle e^{iq \cdot x}\end{aligned}$$

$$\boxed{a_{\mathbf{p}}^{\dagger} \psi^{\dagger}(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} e^{ip \cdot x}}$$

$$\begin{aligned}\boxed{\psi(x) a_{\mathbf{p}}^{\dagger}} &= \langle 0 | \psi(x) a_{\mathbf{p}}^{\dagger} | 0 \rangle \\ &= \int \frac{d^3 q}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{q}})^{1/2}} \langle \mathbf{q} | \mathbf{p} \rangle e^{-iq \cdot x}\end{aligned}$$

$$\boxed{\psi(x) a_{\mathbf{p}}^{\dagger} = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} e^{-ip \cdot x}}$$

$$\begin{aligned}\boxed{b_{\mathbf{p}}^{\dagger} \psi(x)} &= \langle 0 | b_{\mathbf{p}} \psi(x) | 0 \rangle \\ &= \int \frac{d^3 q}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{q}})^{1/2}} \langle \mathbf{p} | \mathbf{q} \rangle e^{iq \cdot x}\end{aligned}$$

$$\boxed{b_{\mathbf{p}}^{\dagger} \psi(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} e^{ip \cdot x}}$$

$$\begin{aligned}\boxed{\psi^{\dagger}(x) b_{\mathbf{p}}^{\dagger}} &= \langle 0 | \psi^{\dagger}(x) b_{\mathbf{p}}^{\dagger} | 0 \rangle \\ &= \int \frac{d^3 q}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{q}})^{1/2}} \langle \mathbf{q} | \mathbf{p} \rangle e^{-iq \cdot x}\end{aligned}$$

$$\boxed{\psi^{\dagger}(x) b_{\mathbf{p}}^{\dagger} = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} e^{-ip \cdot x}}$$

$$\begin{aligned}\boxed{c_{\mathbf{q}} \phi(x)} &= \langle 0 | c_{\mathbf{q}} \psi(x) | 0 \rangle \\ &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_{\mathbf{p}})^{1/2}} \langle \mathbf{q} | \mathbf{p} \rangle e^{ip \cdot x}\end{aligned}$$

$$\boxed{c_{\mathbf{q}} \phi(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_{\mathbf{q}})^{1/2}} e^{iq \cdot x}}$$

$$\begin{aligned}\boxed{\phi(x) c_{\mathbf{q}}^{\dagger}} &= \langle 0 | \phi(x) c_{\mathbf{q}}^{\dagger} | 0 \rangle \\ &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_{\mathbf{p}})^{1/2}} \langle \mathbf{p} | \mathbf{q} \rangle e^{-ip \cdot x}\end{aligned}$$

$$\boxed{\phi(x) c_{\mathbf{q}}^{\dagger} = \frac{1}{(2\pi)^{3/2}} \frac{1}{(2\varepsilon_{\mathbf{q}})^{1/2}} e^{-iq \cdot x}}$$

20.2

Given the function

$$V(\mathbf{r}) = -\frac{g^2}{4\pi|\mathbf{r}|} e^{-\mu|\mathbf{r}|}$$

we can calculate its Fourier transform as

$$\begin{aligned}\tilde{V}(\mathbf{q}) &= \int d^3 r V(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{r}} \\ &= -\frac{g^2}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^1 d(\cos \vartheta) \int_0^\infty dr r^2 \frac{1}{r} e^{iqr \cos \vartheta - \mu r} \\ &= \frac{ig^2}{2q} \int_0^\infty dr e^{-\mu r} (e^{iqr} - e^{-iqr}) \\ &= \frac{ig^2}{2q} \left[\frac{1}{iq - \mu} e^{(iq - \mu)r} \Big|_0^\infty + \frac{1}{iq + \mu} e^{-(iq + \mu)r} \Big|_0^\infty \right] \\ &= -\frac{ig^2}{2q} \left[\frac{1}{iq - \mu} + \frac{1}{iq + \mu} \right] \\ &= -\frac{ig^2}{2q} \frac{iq + \mu + iq - \mu}{-q^2 - \mu^2} \\ \boxed{\tilde{V}(\mathbf{q}) = \frac{-g^2}{\mathbf{q}^2 + \mu^2}}\end{aligned}$$

The Coulomb potential is the massless limit of the Yukawa potential

$$V_{\text{coulomb}}(\mathbf{r}) = \lim_{\mu \rightarrow 0} \left(-\frac{g^2}{4\pi|\mathbf{r}|} e^{-\mu|\mathbf{r}|} \right)$$

Taking the corresponding limit of the Fourier transform, we find

$$\boxed{\tilde{V}_{\text{coulomb}}(\mathbf{q}) = \frac{-g^2}{\mathbf{q}^2}}$$