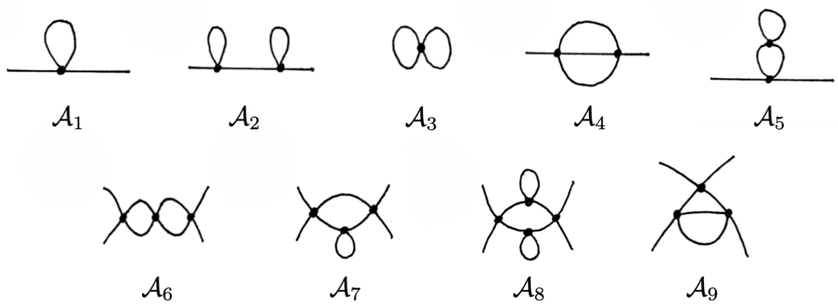
Chapter 19

Expanding the S-matrix: Feynman diagrams

19.1

The amplitudes for the diagrams



are given in momentum space as

$$
\begin{aligned}
\mathcal{A}_1 &= (2\pi)^4\delta(p-q)\frac{-i\lambda}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \\
\mathcal{A}_2 &= (2\pi)^4\delta(p-q)\frac{(-i\lambda)^2}{4}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(p^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)} \\
\mathcal{A}_3 &= \frac{-i\lambda}{8}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)}\int \mathrm{d}^4x \\
\mathcal{A}_4 &= (2\pi)^4\delta(p-q)\frac{(-i\lambda)^2}{6}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)} \\
&\times\frac{i}{((p-k_1-k_2)^2-m^2+i\epsilon)} \\
\mathcal{A}_5 &= (2\pi)^4\delta(p-q)\frac{(-i\lambda)^2}{4}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\left(\frac{i}{(k_1^2-m^2+i\epsilon)}\right)^2\frac{i}{(k_2^2-m^2+i\epsilon)} \\
\mathcal{A}_6 &= (2\pi)^4\delta(p_1+p_2-q_1-q_2)\frac{(-i\lambda)^3}{4}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)} \\
&\times\frac{i}{((p_1+p_2-k_1)^2-m^2+i\epsilon)}\frac{i}{((p_1+p_2-k_2)^2-m^2+i\epsilon)} \\
\mathcal{A}_7 &= (2\pi)^4\delta(p_1+p_2-q_1-q_2)\frac{(-i\lambda)^3}{2}\int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\left(\frac{i}{(k_1^2-m^2+i\epsilon)}\right)^2\frac{i}{(k_2^2-m^2+i\epsilon)} \\
&\times\frac{i}{((p_1+p_2-k_1)^2-m^2+i\epsilon)}
\end{aligned}
$$

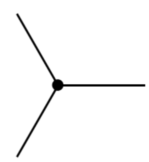
$$\mathcal{A}_8 = (2\pi)^4\delta(p_1+p_2-q_1-q_2)\frac{(-i\lambda)^4}{8}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{d}^4k_3}{(2\pi)^{12}}\left(\frac{i}{(k_1^2-m^2+i\epsilon)}\right)^2$$
$$\times\frac{i}{(k_2^2-m^2+i\epsilon)}\frac{i}{(k_3^2-m^2+i\epsilon)}\left(\frac{i}{((p_1+p_2-k_1)^2-m^2+i\epsilon)}\right)^2$$
$$\mathcal{A}_9 = (2\pi)^4\delta(p_1+p_2-q_1-q_2)\frac{(-i\lambda)^3}{2}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)}$$
$$\times\frac{i}{((p_1-k_1-k_2)^2-m^2+i\epsilon)}\frac{i}{((q_1-k_1-k_2)^2-m^2+i\epsilon)}$$

19.2

The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\eta}{3!}\varphi^3$$

has an interaction vertex given by



For this interaction, we can write the general one-particle S-matrix element as

$$\langle q|S|p\rangle = \langle q|T\left[\exp\left\{\frac{-i\eta}{3!}\int\mathrm{d}^4x\,\varphi(x)\varphi(x)\varphi(x)\right\}\right]|p\rangle$$

Expanding this matrix element to $\mathcal{O}(\eta^2)$ and recalling the relativistic normalization of momentum states

$$|p\rangle = (2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}a^\dagger_{\mathbf{p}}|0\rangle$$
$$\langle q|p\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})$$

we can write

$$\langle q|S|p\rangle = \langle q|p\rangle + \frac{-i\eta}{3!}\int\mathrm{d}^4x\,\langle q|T\Big[\varphi(x)\varphi(x)\varphi(x)\Big]|p\rangle$$
$$+\frac{1}{2!}\left(\frac{-i\eta}{3!}\right)^2\int\mathrm{d}^4x\,\mathrm{d}^4y\,\langle q|T\Big[\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)\Big]|p\rangle$$
$$\langle q|S|p\rangle = (2\pi)^3(4E_{\mathbf{q}}E_{\mathbf{p}})^{1/2}\Bigg(\delta(\mathbf{p}-\mathbf{q}) + \frac{-i\eta}{3!}\int\mathrm{d}^4x\,\langle 0|T\Big[a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)a^\dagger_{\mathbf{p}}\Big]|0\rangle$$
$$+\frac{1}{2!}\left(\frac{-i\eta}{3!}\right)^2\int\mathrm{d}^4x\int\mathrm{d}^4y\,\langle 0|T\Big[a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a^\dagger_{\mathbf{p}}\Big]|0\rangle\Bigg)$$

Using Wick's theorem, all terms proportional to odd powers of η vanish, leaving

$$\langle q|S|p\rangle = (2\pi)^3(4E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}\Bigg[\delta(\mathbf{p}-\mathbf{q})$$
$$+\frac{1}{2!}\left(\frac{-i\eta}{3!}\right)^2\int\mathrm{d}^4x\int\mathrm{d}^4y\sum_{\text{Wick}}\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a^\dagger_{\mathbf{p}}|0\rangle\Bigg]$$

$$
\begin{aligned}
&= (2\pi)^3(4E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}\Big[\delta(\mathbf{p}-\mathbf{q}) \\
&+ \frac{1}{2!}\left(\frac{-i\eta}{3!}\right)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \Big(9\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle \\
&+ 6\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle + 18\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle \\
&+ 36\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle + 36\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle\Big)\Big] \\
&= (2\pi)^3(4E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}\Big[\delta(\mathbf{p}-\mathbf{q}) \\
&+ (-i\eta)^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \Big(\frac{1}{8}\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle \\
&+ \frac{1}{12}\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle + \frac{1}{4}\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle \\
&+ \frac{1}{2}\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle + \frac{1}{2}\,\langle 0|a_{\mathbf{q}}\varphi(x)\varphi(x)\varphi(x)\varphi(y)\varphi(y)\varphi(y)a_{\mathbf{p}}^\dagger|0\rangle\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\langle q|S|p\rangle &= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\left[1 + (-i\eta)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\left(\frac{1}{8}\Delta(0)^2\Delta(x-y) + \frac{1}{12}\Delta(x-y)^3\right)\right] \\
&+ (-i\eta)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\left(\frac{1}{4}\Delta(0)^2 e^{i(q\cdot x - p\cdot y)} + \frac{1}{2}\Delta(x-y)^2 e^{i(q\cdot x-p\cdot y)} + \frac{1}{2}\Delta(0)\Delta(x-y)e^{i(q-p)\cdot x}\right)
\end{aligned}
$$

Transferring to momentum space, we can write

$$
\langle q|S|p\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q}) + \mathcal{A}_1^{(2)} + \mathcal{A}_2^{(2)} + \mathcal{A}_3^{(2)} + \mathcal{A}_4^{(2)} + \mathcal{A}_5^{(2)}
$$

with

$$
\begin{aligned}
\mathcal{A}_1^{(2)} &= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{8}\Delta(0)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,\Delta(x-y) \\
&= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{8}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)\int \mathrm{d}^4z\int \mathrm{d}^4y\int\frac{\mathrm{d}^4k_3}{(2\pi)^4}\tilde{\Delta}(k_3)e^{-ik_3\cdot z} \\
&= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{8}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)\tilde{\Delta}(0)\int \mathrm{d}^4y
\end{aligned}
$$

$$
\boxed{\mathcal{A}_1^{(2)} = (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{8}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{m^2}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)}\int \mathrm{d}^4y}
$$

$$
\begin{aligned}
\mathcal{A}_2^{(2)} &= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{12}\int \mathrm{d}^4x\int \mathrm{d}^4y\,\Delta(x-y)^3 \\
&= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{12}\int \mathrm{d}^4z\int \mathrm{d}^4y\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2\,\mathrm{d}^4k_3}{(2\pi)^{12}}\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)\tilde{\Delta}(k_3-k_1-k_2)e^{-ik_3\cdot z} \\
&= (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{12}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\tilde{\Delta}(k_1)\tilde{\Delta}(k_2)\tilde{\Delta}(-k_1-k_2)\int \mathrm{d}^4y
\end{aligned}
$$

$$
\boxed{\mathcal{A}_2^{(2)} = (2\pi)^3 2E_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{q})\frac{(-i\eta)^2}{12}\int\frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{(k_2^2-m^2+i\epsilon)}\frac{i}{((-k_1-k_2)^2-m^2+i\epsilon)}\int \mathrm{d}^4y}
$$

$$
\begin{aligned}
\mathcal{A}_3^{(2)} &= \frac{(-i\eta)^2}{4}\Delta(0)^2\int \mathrm{d}^4x\int \mathrm{d}^4y\,e^{i(q\cdot x-p\cdot y)} \\
&= \left(\frac{-i\eta}{2}\Delta(0)\int \mathrm{d}^4y\,e^{-ip\cdot y}\right)\left(\frac{-i\eta}{2}\Delta(0)\int \mathrm{d}^4x\,e^{iq\cdot x}\right) \\
&= \left(\frac{-i\eta}{2}\Delta(0)\delta(p)\right)\left(\frac{-i\eta}{2}\Delta(0)\delta(q)\right)
\end{aligned}
$$

$$
\boxed{\mathcal{A}_3^{(2)} = 0}
$$

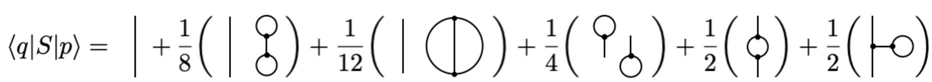
$$\begin{aligned}
\mathcal{A}_4^{(2)} &= \frac{(-i\eta)^2}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y\, \Delta(x-y)^2 e^{i(q\cdot x - p\cdot y)} \\
&= \frac{(-i\eta)^2}{2}\int \mathrm{d}^4z \int \mathrm{d}^4y\, \Delta(z)^2 e^{iq\cdot z} e^{i(q-p)\cdot y} \\
&= (2\pi)^4\delta(p-q)\frac{(-i\eta)^2}{2}\int \mathrm{d}^4z \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8}\tilde{\Delta}(k_1)\tilde{\Delta}(k_2-k_1)e^{-i(k_2-q)z}
\end{aligned}$$

$$\boxed{\mathcal{A}_4^{(2)} = (2\pi)^4\delta(p-q)\frac{(-i\eta)^2}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{i}{(k_1^2-m^2+i\epsilon)}\frac{i}{((q-k_1)^2-m^2+i\epsilon)}}$$

$$\begin{aligned}
\mathcal{A}_5^{(2)} &= \frac{(-i\eta)^2}{2}\Delta(0)\int \mathrm{d}^4x \int \mathrm{d}^4y\, \Delta(x-y) e^{i(q-p)\cdot x} \\
&= \frac{(-i\eta)^2}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\tilde{\Delta}(k_1)\int \mathrm{d}^4z \int \mathrm{d}^4y\, \Delta(z) e^{i(q-p)\cdot z} e^{i(q-p)\cdot y} \\
&= (2\pi)^4\delta(p-q)\frac{(-i\eta)^2}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\tilde{\Delta}(k_1)\int \mathrm{d}^4z \int \frac{\mathrm{d}^4k_2}{(2\pi)^4}\tilde{\Delta}(k_2)e^{i(q-p-k_2)\cdot z}
\end{aligned}$$

$$\boxed{\mathcal{A}_5^{(2)} = (2\pi)^4\delta(p-q)\frac{(-i\eta)^2}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{i}{m^2}\frac{i}{(k_1^2-m^2+i\epsilon)}}$$

The terms in this expansion can be represented diagramatically as



The prefactor of each term is just the reciprocal of the symmetry factor S of the diagram, which can be calculated for this theory via

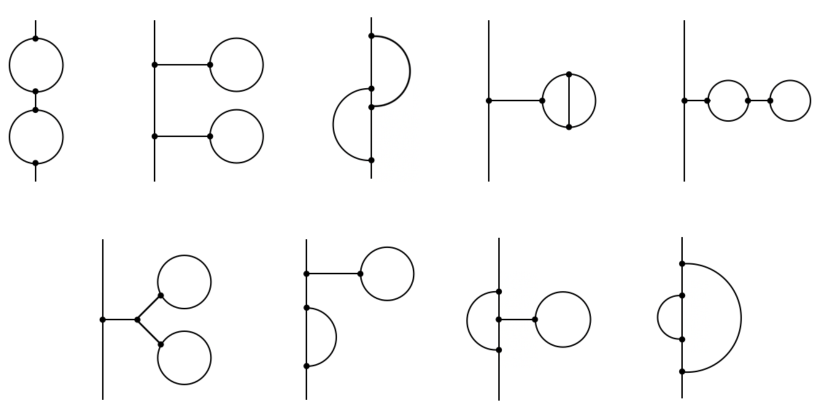
$$S = g2^{\beta}\prod_n (n!)^{\alpha_n}$$

g : the number of permutations of vertices that leave the diagram unchanged with fixed external lines
β : the number of bubbles
α_n : the number of pairs of vertices connected by n identical lines

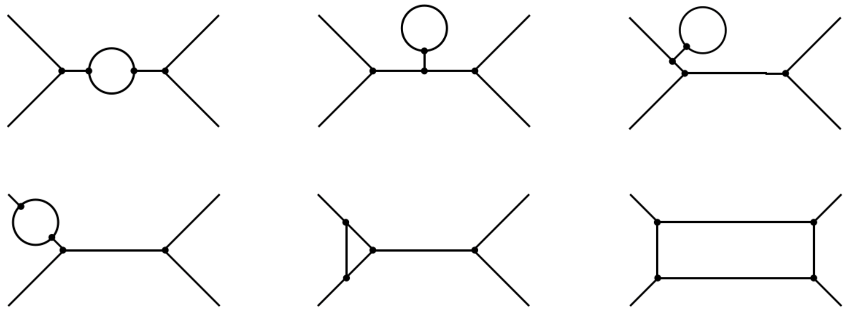
Using this formula, we can confirm for the diagrams above that

$$S_1 = 1, \quad S_2 = 2\cdot 2^2 = 8, \quad S_3 = 2\cdot(3!)^1 = 12, \quad S_4 = 2^2 = 4, \quad S_5 = (2!)^1 = 2, \quad S_6 = 2$$

At $\mathcal{O}(\eta^4)$, the one-particle connected Feynman diagrams which appear are



At $\mathcal{O}(\eta^4)$, the two-particle connected Feynman diagrams which appear are

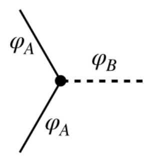


19.3

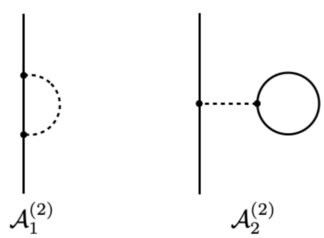
The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_A)^2 - \frac{1}{2}m_A^2\varphi_A^2 + \frac{1}{2}(\partial_\mu \varphi_B)^2 - \frac{1}{2}m_B^2\varphi_B^2 - \frac{g}{2}\varphi_A\varphi_B\varphi_A$$

has an interaction vertex



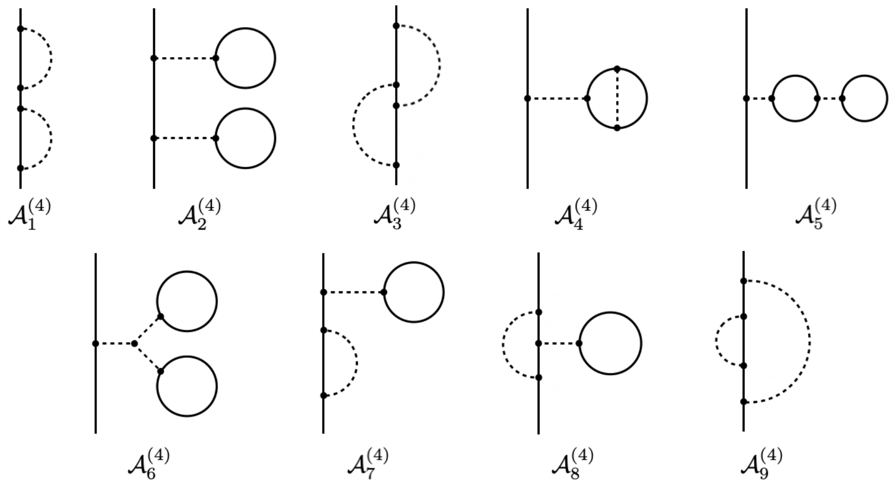
At $\mathcal{O}(g^2)$, the one-particle connected Feynman diagrams which appear in the expansion of $\langle q_A|S|p_A\rangle$ are



These diagrams correspond to the amplitudes

$$\mathcal{A}_1^{(2)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^2 \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \frac{i}{(k_1^2 - m_B^2 + i\epsilon)} \frac{i}{((p_A - k_1)^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_2^{(2)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^2 \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} \frac{i}{m_B^2} \frac{i}{(k_1^2 - m_A^2 + i\epsilon)}$$

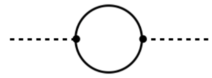
At $\mathcal{O}(g^4)$, the one-particle connected Feynman diagrams which appear in the expansion of $\langle q_A|S|p_A\rangle$ are



These diagrams correspond to the amplitudes

$$\mathcal{A}_1^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \frac{i}{(k_1^2 - m_B + i\epsilon)} \frac{i}{((p_A - k_1)^2 - m_A^2 + i\epsilon)} \frac{i}{(p_A^2 - m_A^2 + i\epsilon)}$$
$$\times \frac{i}{(k_2^2 - m_B^2 + i\epsilon)} \frac{i}{((p_A - k_2)^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_2^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \left(\frac{i}{m_B^2}\right)^2 \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \frac{i}{(k_2^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_3^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \frac{i}{(k_1^2 - m_B^2 + i\epsilon)} \frac{i}{((p_A - k_1)^2 - m_A^2 + i\epsilon)} \frac{i}{(k_2^2 - m_B^2 + i\epsilon)}$$
$$\times \frac{i}{((p_A - k_1 - k_2)^2 - m_A^2 + i\epsilon)} \frac{i}{((p_A - k_2)^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_4^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \frac{i}{m_B^2} \left(\frac{i}{(k_1^2 - m_A^2 + i\epsilon)}\right)^2 \frac{i}{(k_2^2 - m_B^2 + i\epsilon)}$$
$$\times \frac{i}{((k_1 - k_2)^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_5^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \left(\frac{i}{m_B^2}\right)^2 \left(\frac{i}{(k_1^2 - m_A^2 + i\epsilon)}\right)^2 \frac{i}{(k_2^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_6^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \left(\frac{i}{m_B^2}\right)^3 \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \frac{i}{(k_2^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_7^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \frac{i}{m_B^2} \frac{i}{(k_1^2 - m_B^2 + i\epsilon)} \frac{i}{((p_A - k_1)^2 - m_A^2 + i\epsilon)} \frac{i}{(k_2^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_8^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \frac{i}{m_B^2} \frac{i}{(k_1^2 - m_B^2 + i\epsilon)} \left(\frac{i}{((p_A - k_1)^2 - m_A^2 + i\epsilon)}\right)^2$$
$$\times \frac{i}{(k_2^2 - m_A^2 + i\epsilon)}$$
$$\mathcal{A}_9^{(4)} \sim (2\pi)^4\delta(p_A - q_A)(-ig)^4 \int \frac{\mathrm{d}^4k_1\,\mathrm{d}^4k_2}{(2\pi)^8} \frac{i}{(k_1^2 - m_B^2 + i\epsilon)} \left(\frac{i}{((p_A - k_1)^2 - m_A^2 + i\epsilon)}\right)^2 \frac{i}{(k_2^2 - m_B^2 + i\epsilon)}$$
$$\times \frac{i}{((p_A - k_1 - k_1)^2 - m_A^2 + i\epsilon)}$$

At $\mathcal{O}(g^2)$, the one-particle connected Feynman diagram which appears in the expansion of $\langle q_B|S|p_B\rangle$ is

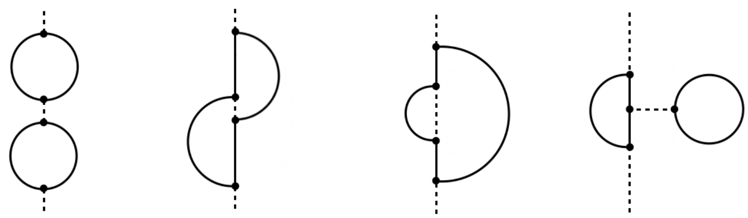


$\mathcal{A}_1^{(2)}$

This diagram corresponds to the amplitude

$$\mathcal{A}_1^{(2)} \sim (2\pi)^4 \delta(p_B - q_B)(-ig)^2 \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \frac{i}{((p_B - k_1)^2 - m_A^2 + i\epsilon)}$$

At $\mathcal{O}(g^4)$, the one-particle connected Feynman diagram which appears in the expansion of $\langle q_B|S|p_B\rangle$ is

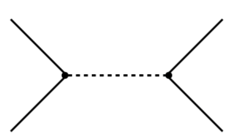


$\mathcal{A}_1^{(4)}$ $\mathcal{A}_2^{(4)}$ $\mathcal{A}_3^{(4)}$ $\mathcal{A}_4^{(4)}$

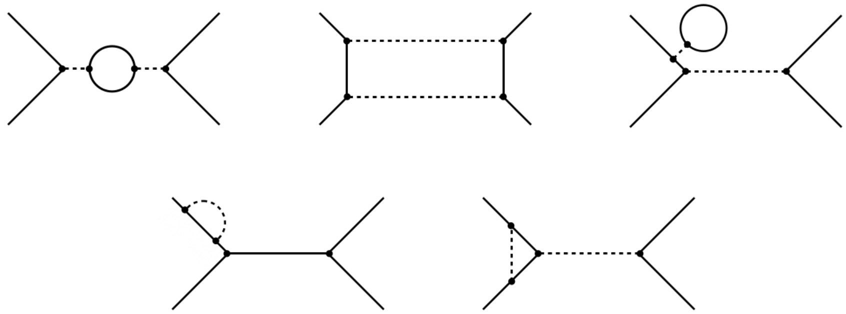
These diagrams correspond to the amplitudes

$$\begin{aligned}
\mathcal{A}_1^{(4)} \sim{}& (2\pi)^4 \delta(p_B - q_B)(-ig)^4 \int \frac{\mathrm{d}^4 k_1\, \mathrm{d}^4 k_2}{(2\pi)^8} \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \frac{i}{((p_B - k_1)^2 - m_A^2 + i\epsilon)} \frac{i}{(p_B^2 - m_B^2 + i\epsilon)} \\
&\times \frac{i}{(k_2^2 - m_A^2 + i\epsilon)} \frac{i}{((p_B - k_2)^2 - m_A^2 + i\epsilon)} \\
\mathcal{A}_2^{(4)} \sim{}& (2\pi)^4 \delta(p_B - q_B)(-ig)^4 \int \frac{\mathrm{d}^4 k_1\, \mathrm{d}^4 k_2}{(2\pi)^8} \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \frac{i}{((p_B - k_1)^2 - m_A^2 + i\epsilon)} \frac{i}{((k_2^2 - m_A^2 + i\epsilon)} \\
&\times \frac{i}{((p_B - k_1 - k_2)^2 - m_B^2 + i\epsilon)} \frac{i}{((p_B - k_2)^2 - m_A^2 + i\epsilon)} \\
\mathcal{A}_3^{(4)} \sim{}& (2\pi)^4 \delta(p_B - q_B)(-ig)^4 \int \frac{\mathrm{d}^4 k_1\, \mathrm{d}^4 k_2}{(2\pi)^8} \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \left(\frac{i}{((p_B - k_1)^2 - m_A^2 + i\epsilon)}\right)^2 \frac{i}{(k_2^2 - m_A^2 + i\epsilon)} \\
&\times \frac{i}{((p_B - k_1 - k_2)^2 - m_B^2 + i\epsilon)} \\
\mathcal{A}_4^{(4)} \sim{}& (2\pi)^4 \delta(p_B - q_B)(-ig)^4 \int \frac{\mathrm{d}^4 k_1\, \mathrm{d}^4 k_2}{(2\pi)^8} \frac{i}{m_B^2} \frac{i}{(k_1^2 - m_A^2 + i\epsilon)} \left(\frac{i}{(p_B - k_1)^2 - m_A^2 + i\epsilon)}\right)^2 \\
&\times \frac{i}{(k_2^2 - m_A^2 + i\epsilon)}
\end{aligned}$$

At $\mathcal{O}(g^2)$, the two-particle connected Feynman diagram which appears in the expansion of $\langle q_{A1} q_{A2}|S|p_{A1} p_{A2}\rangle$ is



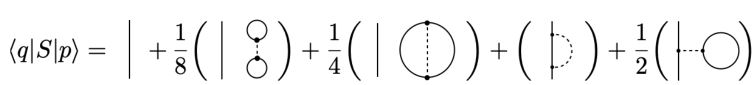
At $\mathcal{O}(g^4)$, the two-particle connected Feynman diagrams which appear in the expansion of $\langle q_{A1} q_{A2}|S|p_{A1} p_{A2}\rangle$ is



We can deduce the rules for the symmetry factors for this theory by expanding the S-matrix and studying the combinatorics as follows

$$
\begin{aligned}
\langle q|S|p\rangle =\ & \langle q|T\exp\left\{-\frac{ig}{2}\int \mathrm{d}^4x\,\varphi_A(x)\varphi_B(x)\varphi_A(x)\right\}|p\rangle \\
=\ & \langle q|p\rangle + \frac{1}{2!}\left(\frac{-ig}{2}\right)^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,T\,\langle q|\varphi_A(x)\varphi_B(x)\varphi_A(x)\varphi_A(y)\varphi_B(y)\varphi_A(x)|p\rangle + \dots \\
=\ & \langle q|p\rangle + (2\pi)^3(4E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}\frac{(-ig)^2}{8}\int \mathrm{d}^4x\,\mathrm{d}^4y \\
& \times\ \langle 0|T\Big[a_{\mathbf{q}}\varphi_A(x)\varphi_B(x)\varphi_A(x)\varphi_A(y)\varphi_B(y)\varphi_A(y)a^\dagger_{\mathbf{p}}\Big]|0\rangle + \dots \\
=\ & \langle q|p\rangle + (2\pi)^3(4E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}\frac{(-ig)^2}{8}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big[\ \langle 0|a_{\mathbf{q}}\varphi_A(x)\varphi_B(x)\varphi_A(x)\varphi_A(y)\varphi_B(y)\varphi_A(y)a^\dagger_{\mathbf{p}}|0\rangle \\
& + 2\,\langle 0|a_{\mathbf{q}}\varphi_A(x)\varphi_B(x)\varphi_A(x)\varphi_A(y)\varphi_B(y)\varphi_A(y)a^\dagger_{\mathbf{p}}|0\rangle \\
& + 8\,\langle 0|a_{\mathbf{q}}\varphi_A(x)\varphi_B(x)\varphi_A(x)\varphi_A(y)\varphi_B(y)\varphi_A(y)a^\dagger_{\mathbf{p}}|0\rangle \\
& + 4\,\langle 0|a_{\mathbf{q}}\varphi_A(x)\varphi_B(x)\varphi_A(x)\varphi_A(y)\varphi_B(y)\varphi_A(y)a^\dagger_{\mathbf{p}}|0\rangle + \dots\Big] + \dots \\
\langle q|S|p\rangle =\ & \langle q|p\rangle + (2\pi)^3(4E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}(-ig)^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\Big[\frac{1}{8}\delta(\mathbf{p}-\mathbf{q})\Delta_A(0)^2\Delta_B(x-y) \\
& + \frac{1}{4}\delta(\mathbf{p}-\mathbf{q})\Delta_A(x-y)^2\Delta_B(x-y) + \Delta_A(x-y)\Delta_B(x-y)e^{i(q_A\cdot x-p_A\cdot y)} \\
& + \frac{1}{2}\Delta_A(0)\Delta_B(x-y)e^{i(q_A-p_A)\cdot x} + \dots\Big] + \dots
\end{aligned}
$$

The corresponding diagrams are



From these diagrams, we can see that the symmetry rules are the same as for the φ^3 theory given in the previous problem.

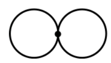
19.4

The double bubble diagram comes at $\mathcal{O}(\lambda)$ in the φ^4 theory in the form

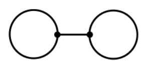
$$\begin{aligned}\mathcal{A} &= \frac{-i\lambda}{4!}\int \mathrm{d}^4x\, 3\,\langle 0|\varphi(x)\varphi(x)\varphi(x)\varphi(x)|0\rangle \\ &= \frac{-i\lambda}{8}\int \mathrm{d}^4x\,\Delta(0)^2\end{aligned}$$

$$\boxed{\mathcal{A} = \frac{-i\lambda}{8}\left[\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\right]^2\int \mathrm{d}^4x}$$

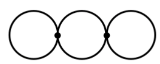
where the right-most term gives an infinite factor of spacetime volume. This factor arises for all vacuum diagrams since if there are n vertices, there are only $n-1$ independent propagators which can connect them. Some examples are shown below



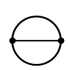
$$\propto \Delta(0)^2\int \mathrm{d}^4x$$



$$\propto \Delta(0)^2\int \mathrm{d}^4z\,\Delta(z)\int \mathrm{d}^4x$$



$$\propto \Delta(0)^2\int \mathrm{d}^4z\,\Delta(z)^2\int \mathrm{d}^4x$$



$$\propto \int \mathrm{d}^4z\,\Delta(z)^3\int \mathrm{d}^4x$$

$$\vdots$$