Chapter 18

The S-matrix

18.1

A spin-1/2 particle in a constant magnetic field subjected to a small, perpendicular perturbation has a Hamiltonian

$$\hat{H} = \gamma B_0 \hat{S}_z + \gamma B_1 \Big(\hat{S}_x \cos(\gamma B_0 t) + \hat{S}_y \sin(\gamma B_0 t) \Big)$$

Defining the natural frequency $\omega \equiv \gamma B_0$, this has the form

$$\hat{H} = \omega \hat{S}_z + \gamma B_1 \Big(\hat{S}_x \cos \omega t + \hat{S}_y \sin \omega t \Big)$$

This Hamiltonian can be easily split into a free part and an interacting part, $\hat{H} = \hat{H}_0 + \hat{H}'$, where

$$\hat{H}_0 = \omega \hat{S}_z, \quad \hat{H}' = \gamma B_1 \Big(\hat{S}_x \cos \omega t + \hat{S}_y \sin \omega t \Big)$$

With this, we can express the Hamiltonian in the interacting picture

$$\hat{H}_I = e^{i\hat{H}_0 t} \hat{H}' e^{-i\hat{H}_0 t}$$

$$\boxed{\hat{H}_I = \gamma B_1 e^{i\omega \hat{S}_z t} \Big(\hat{S}_x \cos \omega t + \hat{S}_y \sin \omega t \Big) e^{-i\omega \hat{S}_z t}}$$

This can be simplified using the definitions of the raising and lowering spin operators $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$

$$\begin{aligned}
\hat{H}_I &= \frac{1}{2} \gamma B_1 e^{i\omega \hat{S}_z t} \Big[\hat{S}_x \big(e^{i\omega t} + e^{-i\omega t} \big) - i\hat{S}_y \big(e^{i\omega t} - e^{-i\omega t} \big) \Big] e^{-i\omega \hat{S}_z t} \\
&= \frac{1}{2} \gamma B_1 e^{i\omega \hat{S}_z t} \Big[e^{i\omega t} \Big(\hat{S}_x - i\hat{S}_y \Big) + e^{-i\omega t} \Big(\hat{S}_x + i\hat{S}_y \Big) \Big] e^{-i\omega \hat{S}_z t} \\
\hat{H}_I &= \frac{1}{2} \gamma B_1 e^{i\omega \hat{S}_z t} \Big(e^{i\omega t} \hat{S}_- + e^{-i\omega t} \hat{S}_+ \Big) e^{-i\omega \hat{S}_z t}
\end{aligned}$$

The raising and lowering operators in the interaction picture are given by

$$\begin{aligned}
e^{i\omega \hat{S}_z t} \hat{S}_+ e^{-i\omega \hat{S}_z t} &= e^{i\omega t} \hat{S}_+ \\
e^{i\omega \hat{S}_z t} \hat{S}_- e^{-i\omega \hat{S}_z t} &= e^{-i\omega t} \hat{S}_-
\end{aligned}$$

which further simplifies the Hamiltonian to

$$\hat{H}_I = \frac{1}{2} \gamma B_1 \Big(e^{i\omega t} e^{-i\omega t} \hat{S}_- + e^{-i\omega t} e^{i\omega t} \hat{S}_+ \Big)$$

$$\boxed{\hat{H}_I = \frac{1}{2} \gamma B_1 \Big(\hat{S}_- + \hat{S}_+ \Big)}$$

With this, the interaction picture time evolution operator can be written simply as

$$\hat{U}_I(t_2, t_1) = T \Bigg[\exp \left(-i \int_{t_1}^{t_2} \mathrm{d}t \, \hat{H}_I \right) \Bigg]$$

$$\boxed{\hat{U}_I(t_2, t_1) = \exp \left(-\frac{i\gamma B_1}{2} \Big(\hat{S}_- + \hat{S}_+ \Big) (t_2 - t_1) \right)}$$

We can now calculate transition amplitudes in the interaction picture. If at time $t = 0$, the state is given by

$$|s, m\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \equiv |\uparrow\rangle$$

then the amplitude that it will be found in the same state at time t is given by

$$\begin{aligned}
\mathcal{A} &= \langle\psi_I(t)|\hat{U}_I(t)|\psi_I(0)\rangle \\
&= \langle\psi(t)|e^{-i\hat{H}_0 t}\hat{U}_I(t)|\psi(0)\rangle \\
&= \langle\uparrow|e^{-i\omega\hat{S}_z t}e^{-i\gamma B_1(\hat{S}_- + \hat{S}_+)t/2}|\uparrow\rangle \\
\mathcal{A} &= e^{-i\omega t/2}\,\langle\uparrow|e^{-i\gamma B_1(\hat{S}_- + \hat{S}_+)t/2}|\uparrow\rangle
\end{aligned}$$

Expanding out the exponential, we find

$$\mathcal{A} = e^{-i\omega t/2}\,\langle\uparrow|1 + \left(\frac{-i\gamma B_1 t}{2}\right)\left(\hat{S}_- + \hat{S}_+\right) + \frac{1}{2!}\left(\frac{-i\gamma B_1 t}{2}\right)^2\left(\hat{S}_- + \hat{S}_+\right)^2 + \ldots|\uparrow\rangle$$

Note that the only combinations of operators which can yield a non-zero matrix element are

$$\hat{S}_+\hat{S}_-,\ \left(\hat{S}_+\hat{S}_-\right)^2,\ \left(\hat{S}_+\hat{S}_-\right)^3,\ \text{etc.}$$

which act as follows

$$\begin{aligned}
\hat{S}_-\,|\uparrow\rangle &= \sqrt{\tfrac{1}{2}\left(1 + \tfrac{1}{2}\right) - \tfrac{1}{2}\left(\tfrac{1}{2} - 1\right)}\,|\downarrow\rangle = |\downarrow\rangle \\
\hat{S}_+\,|\downarrow\rangle &= \sqrt{\tfrac{1}{2}\left(1 + \tfrac{1}{2}\right) - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2} + 1\right)}\,|\uparrow\rangle = |\uparrow\rangle
\end{aligned}$$

Therefore, the terms in the expansion will produce

$$\begin{aligned}
\mathcal{A} &= e^{-i\omega t/2}\left[1 + \frac{1}{2!}\left(\frac{-i\gamma B_1 t}{2}\right)^2 + \frac{1}{4!}\left(\frac{-i\gamma B_1 t}{2}\right)^4 + \ldots\right]\langle\uparrow|\uparrow\rangle \\
\mathcal{A} &= e^{-i\omega t/2}\left[1 - \frac{1}{2!}\left(\frac{\gamma B_1 t}{2}\right)^2 + \frac{1}{4!}\left(\frac{\gamma B_1 t}{2}\right)^4 + \ldots\right]
\end{aligned}$$

Defining the frequency $\Omega \equiv \gamma B_1$, this yields

$$\mathcal{A} = e^{-i\omega t/2}\cos\frac{\Omega t}{2}$$

and therefore a probability of

$$\boxed{|\mathcal{A}|^2 = \cos^2\frac{\Omega t}{2}}$$

As for the amplitude that the particle will be found spin down at time t, we can use the same reasoning as above to compute the answer more quickly. We have the expansion

$$\mathcal{A} = e^{i\omega t/2}\,\langle\downarrow|1 + \left(\frac{-i\gamma B_1 t}{2}\right)\left(\hat{S}_- + \hat{S}_+\right) + \frac{1}{2!}\left(\frac{-i\gamma B_1 t}{2}\right)^2\left(\hat{S}_- + \hat{S}_+\right)^2 + \ldots|\uparrow\rangle$$

Now, the only combinations of operators which will yield a non-zero matrix element are

$$\hat{S}_-,\ \left(\hat{S}_-\hat{S}_+\right)\hat{S}_-,\ \left(\hat{S}_-\hat{S}_+\right)^2\hat{S}_-,\ \text{etc.}$$

Therefore, the expansion will yield

$$\mathcal{A} = e^{i\omega t/2}\left[\left(\frac{-i\Omega t}{2}\right) + \frac{1}{3!}\left(\frac{-i\Omega t}{2}\right)^3 + \ldots\right]\langle\downarrow|\downarrow\rangle = -ie^{i\omega t/2}\sin\frac{\Omega t}{2}$$

with probability

$$\boxed{|\mathcal{A}|^2 = \sin^2\frac{\Omega t}{2}}$$

To calculate $\langle \hat{S}_z \rangle$, we need the explicit form of the state $|\psi(t)\rangle$. Using the results of the previous part, we can construct the state as follows. We know if we start in the state $|\psi(0)\rangle = |{\uparrow}\rangle$, the state will evolve into

$$|\psi(t)\rangle = \cos\frac{\Omega t}{2}\,|{\uparrow}\rangle + i\sin\frac{\Omega t}{2}\,|{\downarrow}\rangle$$

If we had started in the state $|\psi(0)\rangle = |{\downarrow}\rangle$, then by the symmetry of the problem, we would have obtained

$$|\psi(t)\rangle = \cos\frac{\Omega t}{2}\,|{\downarrow}\rangle + i\sin\frac{\Omega t}{2}\,|{\uparrow}\rangle$$

Therefore, if we consider the most general initial state

$$|\psi(0)\rangle = \alpha\,|{\uparrow}\rangle + \beta\,|{\downarrow}\rangle\,, \qquad |\alpha|^2 + |\beta|^2 = 1$$

we have the state as a function of time

$$|\psi(t)\rangle = \left(\alpha\cos\frac{\Omega t}{2} + i\beta\sin\frac{\Omega t}{2}\right)|{\uparrow}\rangle + \left(\beta\cos\frac{\Omega t}{2} + i\alpha\sin\frac{\Omega t}{2}\right)|{\downarrow}\rangle$$

Note these states are in the Schrödinger picture, hence the lack of the phases present in the previous part when working in the interaction picture. Since expectation values are picture-independent, we can work in the Schrödinger picture with no loss of generality. With this, we can calculate

$$\begin{aligned}
\langle \hat{S}_z \rangle &= \langle \psi(t)|\hat{S}_z|\psi(t)\rangle \\
&= \frac{1}{2}\left[\left(\alpha^*\cos\frac{\Omega t}{2} - i\beta^*\sin\frac{\Omega t}{2}\right)\langle{\uparrow}| + \left(\beta^*\cos\frac{\Omega t}{2} - i\alpha^*\sin\frac{\Omega t}{2}\right)\langle{\downarrow}|\right] \\
&\times\left[\left(\alpha\cos\frac{\Omega t}{2} + i\beta\sin\frac{\Omega t}{2}\right)|{\uparrow}\rangle - \left(\beta\cos\frac{\Omega t}{2} + i\alpha\sin\frac{\Omega t}{2}\right)|{\downarrow}\rangle\right] \\
&= \frac{1}{2}\left[\left(\alpha^*\cos\frac{\Omega t}{2} - i\beta^*\sin\frac{\Omega t}{2}\right)\left(\alpha\cos\frac{\Omega t}{2} + i\beta\sin\frac{\Omega t}{2}\right)\right. \\
&\left. -\left(\beta^*\cos\frac{\Omega t}{2} - i\alpha^*\sin\frac{\Omega t}{2}\right)\left(\beta\cos\frac{\Omega t}{2} + i\alpha\sin\frac{\Omega t}{2}\right)\right] \\
&= \frac{1}{2}\left[\left(|\alpha|^2 - |\beta|^2\right)\left(\cos^2\frac{\Omega t}{2} - \sin^2\frac{\Omega t}{2}\right) + i(\alpha^*\beta - \alpha\beta^*)\left(2\sin\frac{\Omega t}{2}\cos\frac{\Omega t}{2}\right)\right]
\end{aligned}$$

$$\boxed{\langle \hat{S}_z \rangle = \frac{1}{2}\left(|\alpha|^2 - |\beta|^2\right)\cos\Omega t + \frac{i}{2}(\alpha^*\beta - \alpha\beta^*)\sin\Omega t}$$

18.2

Using the relationship

$$|\psi_I(\pm\infty)\rangle = |\psi\rangle_{\text{simpleworld}} \equiv |\psi\rangle$$

we can write

$$\begin{aligned}
|\psi_I(+\infty)\rangle &= \hat{S}\,|\psi_I(-\infty)\rangle \\
|\psi_I(+\infty)\rangle &= \hat{S}\,|\psi\rangle
\end{aligned}$$

If the collection of "simpleworld" states forms an orthonormal basis, we can insert a resolution of the identity and obtain

$$|\psi_I(+\infty)\rangle = \left(\sum_\phi |\phi\rangle\langle\phi|\right)\hat{S}\,|\psi\rangle$$

$$\boxed{|\psi_I(+\infty)\rangle = \sum_\phi \langle\phi|\hat{S}|\psi\rangle\,|\phi\rangle}$$

18.3

For a string of bosonic operators which are time-independent, the string is identical to the time-ordering of the string

$$\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}} = T\left[\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}}\right]$$

However, we can now implement Wick's theorem and write

$$\begin{aligned}
\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}} &= N\left[\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}}\right] \\
&= \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{k}} + \left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right]\hat{a}_{\mathbf{k}} + \left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{k}}\right]\hat{a}^\dagger_{\mathbf{q}} + \left[\hat{a}^\dagger_{\mathbf{q}}, \hat{a}_{\mathbf{k}}\right]\hat{a}_{\mathbf{p}} \\
&\boxed{\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\delta(\mathbf{p}-\mathbf{q}) - \hat{a}_{\mathbf{p}}\delta(\mathbf{q}-\mathbf{k})}
\end{aligned}$$

18.4

The string of bosonic operators can be written using standard Bose commutation relations as

$$\begin{aligned}
\hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger &= \hat{b}\hat{g}\left(\hat{b}^\dagger\hat{b}+1\right)\hat{b}^\dagger \\
&= \hat{b}\hat{b}^\dagger\hat{g}\hat{b}\hat{b}^\dagger + (\hat{b}^\dagger\hat{b}+1)\hat{g} \\
&= \left(\hat{b}^\dagger\hat{b}+1\right)\hat{g}\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}\hat{g} + \hat{g} \\
&= \hat{b}^\dagger\hat{b}\hat{g}\left(\hat{b}^\dagger\hat{b}+1\right) + \hat{g}\left(\hat{b}^\dagger\hat{b}+1\right) + \hat{b}^\dagger\hat{b}\hat{g} + \hat{g} \\
&= \hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{g}\hat{b} + 3\hat{b}^\dagger\hat{b}\hat{g} + 2\hat{g} \\
&= \hat{b}^\dagger\left(\hat{b}^\dagger\hat{b}+1\right)\hat{g}\hat{b} + 3\hat{b}^\dagger\hat{b}\hat{g} + 2\hat{g} \\
&\boxed{\hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger = \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}\hat{g} + 4\hat{b}^\dagger\hat{b}\hat{g} + 2\hat{g}}
\end{aligned}$$

Using Wick's theorem, we can normal order this string as follows

$$\begin{aligned}
\hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger &= N\left[\hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger + \hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger\right] \\
&\boxed{\hat{b}\hat{g}\hat{b}\hat{b}^\dagger\hat{b}^\dagger = \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}\hat{g} + 4\hat{b}^\dagger\hat{b}\hat{g} + 2\hat{g}}
\end{aligned}$$

which is the same result as normal ordering the string by hand.

18.5

Given the VEV of fermionic operators, we can apply Wick's theorem as follows

$$\begin{aligned}
\langle 0|\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}|0\rangle &= \langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}\right]|0\rangle \\
&= \langle 0|N\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1} + \hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1} + \hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}\right]|0\rangle
\end{aligned}$$

$$\boxed{\langle 0|\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\hat{c}_{\mathbf{p}_1}|0\rangle = -\langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}_{\mathbf{p}_2}\right]|0\rangle\langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_1}\right]|0\rangle + \langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_1-\mathbf{q}}\hat{c}_{\mathbf{p}_1}\right]|0\rangle\langle 0|T\left[\hat{c}^\dagger_{\mathbf{p}_2+\mathbf{q}}\hat{c}_{\mathbf{p}_2}\right]|0\rangle}$$