

# Chapter 17

## Propagators and fields

### 17.1

---

Given the retarded Green's function for a free particle in a momentum-time representation

$$G_0^+(\mathbf{p}, t_x; \mathbf{q}, t_y) = \theta(t_x - t_y) \langle 0 | a_{\mathbf{p}}(t_x) a_{\mathbf{q}}^\dagger(t_y) | 0 \rangle$$

we can remove the time-dependence from the operators

$$G_0^+(\mathbf{p}, t_x; \mathbf{q}, t_y) = \theta(t_x - t_y) \langle 0 | e^{iHt_x} a_{\mathbf{p}} e^{-iHt_x} e^{iHt_y} a_{\mathbf{q}}^\dagger e^{-iHt_y} | 0 \rangle$$

Since this is the non-interacting vacuum, we can ensure through normal-ordering that

$$H | 0 \rangle = 0$$

and therefore write

$$\begin{aligned} G_0^+(\mathbf{p}, t_x; \mathbf{q}, t_y) &= \theta(t_x - t_y) \langle 0 | a_{\mathbf{p}} e^{-iHt_x} e^{iHt_y} a_{\mathbf{q}}^\dagger | 0 \rangle \\ &= \theta(t_x - t_y) \langle \mathbf{p} | e^{-iHt_x} e^{iHt_y} | \mathbf{q} \rangle \\ &= \theta(t_x - t_y) e^{-i(E_{\mathbf{p}} t_x - E_{\mathbf{q}} t_y)} \langle \mathbf{p} | \mathbf{q} \rangle \\ \boxed{G_0^+(\mathbf{p}, t_x; \mathbf{q}, t_y) = \theta(t_x - t_y) e^{-i(E_{\mathbf{p}} t_x - E_{\mathbf{q}} t_y)} \delta^{(3)}(\mathbf{p} - \mathbf{q})} \end{aligned}$$

### 17.2

---

We can confirm that the free scalar propagator is the Green's function of the  $(1+1)$ -dimensional Klein-Gordon equation

$$\left( \frac{\partial^2}{\partial(x^0)^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) \Delta(x, y) = -i\delta^{(2)}(x - y)$$

where

$$\Delta(x, y) = \langle 0 | T\phi(x^0, x) \phi^\dagger(y^0, y) | 0 \rangle = \int \frac{d^2 p}{(2\pi)^2} \frac{ie^{-iE_p(x^0-y^0)+ip(x-y)}}{E_p^2 - p^2 - m^2 + i\epsilon}$$

as follows

$$\begin{aligned} \left( \frac{\partial^2}{\partial(x^0)^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) \Delta(x, y) &= \left( \frac{\partial^2}{\partial(x^0)^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) \int \frac{d^2 p}{(2\pi)^2} \frac{ie^{-iE_p(x^0-y^0)+ip(x-y)}}{E_p^2 - p^2 - m^2 + i\epsilon} \\ &= -i \int \frac{d^2 p}{(2\pi)^2} \frac{E_p^2 - p^2 - m^2}{E_p^2 - p^2 - m^2 + i\epsilon} e^{-iE_p(x^0-y^0)+ip(x-y)} \\ &= -i \int \frac{d^2 p}{(2\pi)^2} e^{-iE_p(x^0-y^0)+ip(x-y)} \\ \boxed{\left( \frac{\partial^2}{\partial(x^0)^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) \Delta(x, y) = -i\delta^{(2)}(x - y)} \end{aligned}$$

## 17.3

---

For the scalar field theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2} \phi(x)^2$$

we can write the field in momentum space as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \partial_\mu \int \frac{d^4 p}{(2\pi)^4} \tilde{\phi}(p) e^{ip \cdot x} \partial^\mu \int \frac{d^4 q}{(2\pi)^4} \tilde{\phi}(q) e^{iq \cdot x} - \frac{m^2}{2} \int \frac{d^4 p d^4 q}{(2\pi)^8} \tilde{\phi}(p) \tilde{\phi}(q) e^{i(p+q) \cdot x} \\ \mathcal{L} &= \frac{1}{2} \int \frac{d^4 p d^4 q}{(2\pi)^8} \tilde{\phi}(p) (-p_\mu q^\mu - m^2) \tilde{\phi}(q) e^{i(p+q) \cdot x}\end{aligned}$$

Using this representation, the action has the form

$$\begin{aligned}S &= \int d^4 x \mathcal{L} \\ &= \frac{1}{2} \int \frac{d^4 p d^4 q}{(2\pi)^4} \tilde{\phi}(p) (-p_\mu q^\mu - m^2) \tilde{\phi}(q) \delta^{(4)}(p + q) \\ S &= \boxed{\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \tilde{\phi}(-p) (p^2 - m^2) \tilde{\phi}(p)}\end{aligned}$$

We can identify the free propagator as  $(i/2)$  times the inverse of the quadratic term between the fields (with an additional  $i\epsilon$  for safety)

$$\boxed{\tilde{G}_0(p) = \frac{i}{p^2 - m^2 + i\epsilon}}$$

## 17.4

---

The quantum simple harmonic oscillator has Lagrangian

$$L = \frac{m}{2} \dot{q}^2 - \frac{1}{2} m \omega_0^2 q^2$$

In momentum space, this reads

$$L = \frac{m}{2} \int \frac{d\omega d\omega'}{(2\pi)^2} \tilde{q}_\omega (-\omega\omega' - \omega_0^2) \tilde{q}_{\omega'} e^{i(\omega+\omega')t}$$

with a corresponding action

$$\begin{aligned}S &= \int dt L \\ &= \frac{m}{2} \int \frac{d\omega d\omega'}{(2\pi)^2} \tilde{q}_\omega (-\omega\omega' - \omega_0^2) \tilde{q}_{\omega'} \delta(\omega + \omega') \\ S &= \frac{m}{2} \int \frac{d\omega}{2\pi} \tilde{q}_{-\omega} (\omega^2 - \omega_0^2) \tilde{q}_\omega\end{aligned}$$

With this we can read off the propagator in momentum space

$$\boxed{\tilde{G}(\omega) = \left(\frac{1}{m}\right) \frac{i}{\omega^2 - \omega_0^2 + i\epsilon}}$$

## 17.5

---

For a system with Lagrangian

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi(x)}{\partial x} \right)^2 + \frac{m^2}{2} \phi(x)^2$$

we can discretize the field as

$$\phi(x) \rightarrow \phi_j = \frac{1}{\sqrt{Na}} \sum_p \tilde{\phi}_p e^{ipja}$$

where  $N$  is the number of lattice sites,  $a$  the lattice spacing, and  $j$  the site label. In this form, the derivative becomes

$$\frac{\partial \phi(x)}{\partial x} \rightarrow \frac{\phi_{j+1} - \phi_j}{a} = \frac{1}{a\sqrt{Na}} \sum_p \tilde{\phi}_p (e^{ip(j+1)a} - e^{ipja})$$

Assuming periodic boundary conditions on the lattice, we can write the square of the derivative as

$$\begin{aligned} \left( \frac{\phi_{j+1} - \phi_j}{a} \right)^2 &= \frac{1}{Na^3} \sum_{p,q} \tilde{\phi}_p \tilde{\phi}_q (e^{ip(j+1)a} - e^{ipja})(e^{iq(j+1)a} - e^{iqja}) \\ &= \frac{1}{Na^3} \sum_{p,q} \tilde{\phi}_p \tilde{\phi}_q (e^{i(p+q)(j+1)a} + e^{i(p+q)ja} - e^{i(p+q)ja} e^{ipa} - e^{i(p+q)ja} e^{iqa}) \\ \left( \frac{\phi_{j+1} - \phi_j}{a} \right)^2 &= \frac{1}{Na^3} \sum_{p,q} \tilde{\phi}_p \tilde{\phi}_q e^{i(p+q)ja} (2 - e^{ipa} - e^{iqa}) \end{aligned}$$

This yields a discretized Lagrangian

$$\mathcal{L} = \frac{1}{Na} \sum_{p,q} \tilde{\phi}_p \left( \frac{2 - e^{ipa} - e^{iqa}}{2a^2} + \frac{m^2}{2} \right) \tilde{\phi}_q e^{i(p+q)ja}$$

As for the action, the integral becomes

$$\int dx \rightarrow Na \sum_j$$

which is consistent with the normalization of the field. Therefore, we have

$$\begin{aligned} S &= \frac{1}{2} \sum_j \sum_{p,q} \tilde{\phi}_p \left( \frac{2 - e^{ipa} - e^{iqa}}{a^2} + m^2 \right) \tilde{\phi}_q e^{i(p+q)ja} \\ &= \frac{1}{2} \sum_{p,q} \tilde{\phi}_p \left( \frac{2 - e^{ipa} - e^{iqa}}{a^2} + m^2 \right) \tilde{\phi}_q \delta_{p,-q} \\ &= \frac{1}{2} \sum_p \tilde{\phi}_{-p} \left( \frac{2 - e^{ipa} - e^{-ipa}}{a^2} + m^2 \right) \tilde{\phi}_p \\ S &= \boxed{\frac{1}{2} \sum_p \tilde{\phi}_{-p} \left( \frac{2}{a^2} - \frac{2}{a^2} \cos pa + m^2 \right) \tilde{\phi}_p} \end{aligned}$$

which yields a momentum-space propagator

$$\boxed{\tilde{G}(p) = \frac{i}{\frac{2}{a^2}(1 - \cos pa) + m^2}}$$

As for the  $(1+1)$ -dimensional elastic string with Lagrangian

$$\mathcal{L} = \frac{1}{2} [(\partial_0 \phi)^2 - (\partial_1 \phi)^2]$$

we can similarly discretize the field as

$$\phi(x, t) \rightarrow \phi_j(t) = \frac{1}{\sqrt{Na}} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_p(\omega) e^{-i\omega t} e^{ipja}$$

With this, the derivatives become

$$\begin{aligned} \partial_0 \phi &\rightarrow \partial_0 \phi_j(t) = \frac{-i}{\sqrt{Na}} \sum_p \int \frac{d\omega}{2\pi} \omega \tilde{\phi}_p(\omega) e^{-i\omega t} e^{ipja} \\ \partial_1 \phi &\rightarrow \frac{\phi_{j+1}(t) - \phi_j(t)}{a} = \frac{1}{\sqrt{Na}} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_p(\omega) e^{-i\omega t} \left( \frac{e^{ip(j+1)a} - e^{ipja}}{a} \right) \end{aligned}$$

the square of which can be written as (again assuming periodic boundary conditions)

$$\begin{aligned} (\partial_0 \phi_j(t))^2 &= \frac{-1}{Na} \sum_{p,q} \int \frac{d\omega d\omega'}{(2\pi)^2} \omega \omega' \tilde{\phi}_p(\omega) \tilde{\phi}_q(\omega') e^{-i(\omega+\omega')t} e^{i(p+q)ja} \\ \left( \frac{\phi_{j+1}(t) - \phi_j(t)}{a} \right)^2 &= \frac{1}{Na^3} \sum_{p,q} \int \frac{d\omega d\omega'}{(2\pi)^2} \tilde{\phi}_p(\omega) \tilde{\phi}_q(\omega') e^{-i(\omega+\omega')t} e^{i(p+q)ja} (2 - e^{ipa} - e^{ipa}) \end{aligned}$$

Therefore, the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2Na} \sum_{p,q} \int \frac{d\omega d\omega'}{(2\pi)^2} \tilde{\phi}_p(\omega) \left( \omega \omega' + \frac{2 - e^{ipa} - e^{ipa}}{a^2} \right) \tilde{\phi}_q(\omega') e^{-i(\omega+\omega')t} e^{i(p+q)ja}$$

The form of the action follows as

$$\begin{aligned} S &= Na \sum_j \int dt \mathcal{L} \\ &= -\frac{1}{2} \sum_{p,q} \int \frac{d\omega d\omega'}{2\pi} \tilde{\phi}_p(\omega) \left( \omega \omega' + \frac{2 - e^{ipa} - e^{ipa}}{a^2} \right) \tilde{\phi}_q(\omega') \delta(\omega + \omega') \delta_{p,-q} \\ &= \frac{1}{2} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_{-p}(-\omega) \left( \omega^2 - \frac{2 - e^{ipa} - e^{-ipa}}{a^2} \right) \tilde{\phi}_p(\omega) \\ S &= \frac{1}{2} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_{-p}(-\omega) \left( \omega^2 - \frac{2}{a^2} (1 - \cos pa) \right) \tilde{\phi}_p(\omega) \end{aligned}$$

Defining  $\omega_0^2 = \frac{2}{a^2}$ , we can read off the momentum-frequency space propagator

$$\tilde{G}(\omega, p) = \frac{i}{\omega^2 - \omega_0^2 (1 - \cos pa)}$$