Chapter 16

Propagators and Green's functions

16.1

For this potential, the wave function vanishes outside the interval $x \in (0, a)$, and inside satisfies

$$H|\phi\rangle = E|\phi\rangle \;\rightarrow\; -\frac{1}{2m}\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = E\phi(x)$$

Defining $k = \sqrt{2mE}$, this equation has normalized solutions

$$\boxed{\phi_n(x) = \sqrt{\frac{2}{a}}\sin(k_n x)}$$

where k_n takes the discrete values $k_n = \dfrac{n\pi}{a}$, $n \in \mathbb{Z}^+$. This yields discrete energy levels

$$\boxed{E_n = \frac{\pi^2 n^2}{2ma^2}}$$

The general form of a momentum-time representation of the retarded Green's function is given by

$$\begin{aligned}
G^+(q, t_x; p, t_y) &= \theta(t_x - t_y)\,\langle q|e^{-iH(t_x - t_y)}|p\rangle \\
&= \theta(t_x - t_y)e^{-iE_p(t_x - t_y)}\delta(p - q) \\
G^+(p; t_x, t_y) &= \theta(t_x - t_y)e^{-iE_p(t_x - t_y)}
\end{aligned}$$

Using the fact that the momentum and energy come in discrete levels, we can write this as

$$\boxed{G^+(n; t_x, t_y) = \theta(t_x - t_y)e^{-i\left(\frac{n^2\pi^2}{2ma^2}\right)(t_x - t_y)}}$$

Lastly, we can obtain a momentum-energy representation by performing a Fourier transform in time (shifting the energies infinitesimally into the complex plane for convergence)

$$\begin{aligned}
G^+(n, \omega) &= \int_{-\infty}^{\infty} \mathrm{d}t_x\, G^+(n; t_x, t_y)e^{i\omega(t_x - t_y)} \\
&= \int_{t_y}^{\infty} \mathrm{d}t_x\, e^{i\left(\omega - \frac{n^2\pi^2}{2ma^2} + i\epsilon\right)(t_x - t_y)} \\
&= \frac{1}{i\left(\omega - \frac{n^2\pi^2}{2ma^2} + i\epsilon\right)}\; e^{i\left(\omega - \frac{n^2\pi^2}{2ma^2} + i\epsilon\right)(t_x - t_y)}\Bigg|_{t_y}^{\infty} \\
\boxed{G^+(n, \omega) = \frac{i}{\omega - \frac{n^2\pi^2}{2ma^2} + i\epsilon}}
\end{aligned}$$

16.2

Starting with the position-time representation of the retarded Green's function

$$G_0^+(t_x, x; t_y, y) = \theta(t_x - t_y) \sum_n \phi_n(x)\phi_n^*(y) e^{-iE_n(t_x - t_y)}$$

we can switch to a position-energy representation by performing a Fourier transform in time with a damping factor as follows

$$\begin{aligned} G_0^+(x, y; E) &= \int_{-\infty}^{\infty} \mathrm{d}t_x \, G_0^+(t_x, x; t_y, y) e^{iE(t_x - t_y)} e^{-\epsilon(t_x - t_y)} \\ &= \sum_n \phi_n(x)\phi_n^*(y) \int_{t_y}^{\infty} \mathrm{d}t_x \, e^{i(E - E_n + i\epsilon)(t_x - t_y)} \\ &= \sum_n \phi_n(x)\phi_n^*(y) \, \frac{1}{i(E - E_n + i\epsilon)} e^{i(E - E_n + i\epsilon)(t_x - t_y)} \Big|_{t_y}^{\infty} \end{aligned}$$

$$\boxed{G_0^+(x, y; E) = \sum_n \frac{i\phi_n(x)\phi_n^*(y)}{E - E_n + i\epsilon}}$$

However, we can perform similar calculations without the damping factor by using the following representation of the heaviside function

$$\theta(t) = i \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \frac{e^{-izt}}{z + i\epsilon}$$

Starting from the momentum-time representation of the Green's function

$$G_0^+(p, t) = \theta(t) e^{-iE_p t}$$

we can obtain the momentum-energy representation through a Fourier transform in time

$$\begin{aligned} G_0^+(p, E) &= \int_{-\infty}^{\infty} \mathrm{d}t \, G_0^+(p, t) e^{iEt} \\ &= i \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \frac{1}{z + i\epsilon} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i(E - E_p - z)t} \\ &= i \int_{-\infty}^{\infty} \mathrm{d}z \, \frac{1}{z + i\epsilon} \delta(E - E_p - z) \end{aligned}$$

$$\boxed{G_0^+(p, E) = \frac{i}{E - E_p + i\epsilon}}$$

16.3

For a one-dimension driven harmonic oscillator, the equation of motion is given by

$$m\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right) A(t - u) = f(t)$$

Considering a driving force of the form $f(t) = \tilde{F}(\omega) e^{-i\omega(t-u)}$. The particular solution can be found by inserting the ansatz

$$A(t - u) = \mathcal{A}(\omega) e^{-i\omega(t-u)}$$

With this, we obtain the condition for the amplitude

$$\begin{aligned} -m(\omega^2 - \omega_0^2)\mathcal{A}(\omega) e^{-i\omega(t-u)} &= \tilde{F}(\omega) e^{-i\omega(t-u)} \\ \mathcal{A}(\omega) &= -\frac{\tilde{F}(\omega)}{m} \frac{1}{\omega^2 - \omega_0^2} \end{aligned}$$

Therefore, the general solution is given by

$$\boxed{A(t-u) = -\frac{\tilde{F}(\omega)}{m}\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2} + B(t)}$$

where $B(t)$ satisfies the homogeneous equation

$$m\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)B(t) = 0$$

The Green's function for this differential equation is the solution to

$$m\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)G(t,u) = \delta(t-u)$$

This resembles the driven differential equation if we represent the delta function as an integral

$$m\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)G(t,u) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega(t-u)}$$

We can solve this by first finding the Fourier transform of the Green's function. Doing this, we obtain

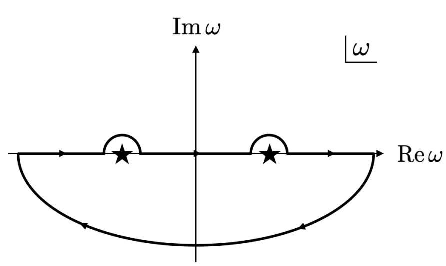
$$\begin{aligned} m\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{G}(\omega)e^{-i\omega(t-u)} &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega(t-u)} \\ \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\Big(m\big(-\omega^2+\omega_0^2\big)\tilde{G}(\omega)\Big)e^{-i\omega(t-u)} &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega(t-u)} \\ \tilde{G}(\omega) &= -\frac{1}{m}\frac{1}{\omega^2-\omega_0^2} \end{aligned}$$

Fourier transforming back, we obtain

$$G(t,u) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\tilde{G}(\omega)e^{-i\omega(t-u)}$$

$$\boxed{G(t,u) = -\frac{1}{m}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega(t-u)}}{\omega^2-\omega_0^2}}$$

Assuming that $t \geq 0$ for the dynamics of the problem, the initial conditions $G(0,u) = \dot{G}(0,u) = 0$ are equivalent to requiring causality in $G(t,u)$. This can be thought of as $G(0,u) = 0$ enforcing that the function vanish at $t = 0$, and $\dot{G}(0,u) = 0$ enforcing that any flow from $t < 0$ vanish. With this, the expression for $G^+(t,u)$ is most easily obtained through complex analysis. With the poles of the function at $\omega = \pm\omega_0$ and $t-u > 0$, we can close the contour in the lower half of the complex $\omega-$plane, and deform the contour up and around the poles so as to include them in the closed contour



Applying Cauchy's residue theorem, we find

$$G^+(t,u) = -\frac{1}{m}\oint \frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega(t-u)}}{(\omega+\omega_0)(\omega-\omega_0)}$$
$$= \frac{i}{m}\left[\frac{e^{-i\omega(t-u)}}{\omega-\omega_0}\bigg|_{\omega=-\omega_0} + \frac{e^{-i\omega(t-u)}}{\omega+\omega_0}\bigg|_{\omega=\omega_0}\right]$$
$$\frac{1}{m\omega_0}\frac{e^{i\omega_0(t-u)} - e^{-i\omega_0(t-u)}}{2i}$$
$$\boxed{G^+(t,u) = \frac{1}{m\omega_0}\sin\omega_0(t-u)}$$

Lastly, for a driving force $f(t) = F_0\sin\omega_0 t$, we can obtain the amplitude as a function of time through the relation

$$A(t) = \int_0^t \mathrm{d}u\, G^+(t,u)f(u)$$
$$= \frac{F_0}{m\omega_0}\int_0^t \mathrm{d}u\sin(\omega_0(t-u))\sin(\omega_0 u)$$
$$\boxed{A(t) = \frac{F_0}{2m\omega_0^2}\Big(\sin(\omega_0 t) - \omega_0 t\cos(\omega_0 t)\Big)}$$

16.4

Considering the Green's function for the Helmholtz differential operator

$$(\nabla^2 + \mathbf{k}^2)G_{\mathbf{k}}(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$$

we can move to momentum space as follows

$$(\nabla^2+\mathbf{k}^2)\int\frac{\mathrm{d}^3q}{(2\pi)^3}\tilde{G}_{\mathbf{k}}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}} = \int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{x}}$$
$$\int\frac{\mathrm{d}^3q}{(2\pi)^3}(-\mathbf{q}^2+\mathbf{k}^2)\tilde{G}_{\mathbf{k}}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}} = \int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot\mathbf{x}}$$
$$(-\mathbf{q}^2+\mathbf{k}^2)\tilde{G}_{\mathbf{k}}(\mathbf{q}) = 1$$
$$\boxed{\tilde{G}_{\mathbf{k}}(\mathbf{q}) = \frac{1}{\mathbf{k}^2-\mathbf{q}^2}}$$

In position space, the Green's function for "outgoing" waves is given by

$$G^+_{\mathbf{k}}(\mathbf{x}) = -\frac{1}{4\pi}\frac{e^{i|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|}$$

This can be confirmed by taking the Fourier transform and making connection with the previous result. Employing a damping factor, we can write the Fourier transform as

$$\tilde{G}^+_{\mathbf{k}}(\mathbf{q}) = \int \mathrm{d}^3x\, G^+_{\mathbf{k}}(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}e^{-\epsilon|\mathbf{x}|}$$
$$= -\frac{1}{4\pi}\int\frac{\mathrm{d}^3x}{|\mathbf{x}|}e^{i(|\mathbf{k}|+i\epsilon)|\mathbf{x}|}e^{i|\mathbf{q}||\mathbf{x}|\cos\vartheta}$$
$$= -\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\int_0^\infty\frac{\mathrm{d}x\,x^2}{x}e^{i(|\mathbf{k}|+i\epsilon)x}e^{i|\mathbf{q}|x\cos\vartheta}$$
$$= -\frac{1}{2}\int_0^\infty \mathrm{d}x\,xe^{i(|\mathbf{k}|+i\epsilon)x}\frac{1}{i|\mathbf{q}|x}\Big(e^{i|\mathbf{q}|x} - e^{-i|\mathbf{q}|x}\Big)$$
$$= -\frac{1}{|\mathbf{q}|}\int_0^\infty \mathrm{d}x\,e^{i(|\mathbf{k}|+i\epsilon)x}\sin(|\mathbf{q}|x)$$
$$\tilde{G}^+_{\mathbf{k}}(\mathbf{q}) = -\frac{1}{|\mathbf{q}|}\frac{|\mathbf{q}|}{|\mathbf{q}|^2-(|\mathbf{k}|+i\epsilon)^2}$$

Since ϵ is an infinitesimal and $r\epsilon = \epsilon \ \ \forall r \in \mathbb{R}$, this can be rewritten as

$$\boxed{\tilde{G}^+_{\mathbf{k}}(\mathbf{q}) = \frac{1}{\mathbf{k}^2 - \mathbf{q}^2 + i\epsilon}}$$

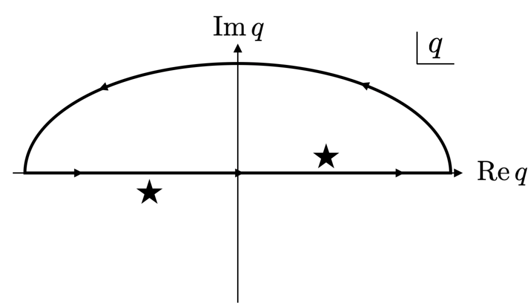
We can indeed go backwards and take the inverse Fourier transform of our result, yielding

$$\begin{aligned}
G^+_{\mathbf{k}}(\mathbf{x}) &= \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G^+_{\mathbf{k}}(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= -\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{e^{-i\mathbf{q}\cdot\mathbf{x}}}{\mathbf{q}^2 - \mathbf{k}^2 - i\epsilon} \\
&= \frac{-1}{(2\pi)^2} \int_{-1}^{1} \mathrm{d}(\cos\vartheta) \int_0^\infty \mathrm{d}q\, q^2 \frac{e^{-iq|\mathbf{x}|\cos\vartheta}}{q^2 - \mathbf{k}^2 - i\epsilon} \\
&= \frac{-1}{4\pi^2} \int_0^\infty \mathrm{d}q\, \frac{q^2}{-iq|\mathbf{x}|} \frac{1}{q^2 - \mathbf{k}^2 - i\epsilon} \left(e^{-iq|\mathbf{x}|} - e^{iq|\mathbf{x}|} \right) \\
G^+_{\mathbf{k}}(\mathbf{x}) &= \frac{i}{4\pi^2 |\mathbf{x}|} \int_{-\infty}^{\infty} \mathrm{d}q\, \frac{q e^{iq|\mathbf{x}|}}{q^2 - \mathbf{k}^2 - i\epsilon}
\end{aligned}$$

A natural factorization of the denominator yields

$$G^+_{\mathbf{k}}(\mathbf{x}) = \frac{i}{4\pi^2 |\mathbf{x}|} \int_{-\infty}^{\infty} \mathrm{d}q\, \frac{q e^{iq|\mathbf{x}|}}{(q - (|\mathbf{k}| + i\epsilon))(q + (|\mathbf{k}| - i\epsilon))}$$

Since $|\mathbf{x}| > 0$, we can close the contour in the upper half of the complex-q plane as shown below



This yields

$$\begin{aligned}
G^+_{\mathbf{k}}(\mathbf{x}) &= \frac{i}{4\pi^2 |\mathbf{x}|} \oint \mathrm{d}q\, \frac{q e^{iq|\mathbf{x}|}}{(q - (|\mathbf{k}| + i\epsilon))(q + (|\mathbf{k}| - i\epsilon))} \\
&= -\frac{1}{2\pi|\mathbf{x}|} \left. \frac{q e^{iq|\mathbf{x}|}}{q + (|\mathbf{k}| - i\epsilon)} \right|_{q = |\mathbf{k}| + i\epsilon} \\
G^+_{\mathbf{k}}(\mathbf{x}) &= -\frac{1}{4\pi} \frac{e^{i|\mathbf{k}||\mathbf{x}|} e^{-\epsilon|\mathbf{x}|}}{|\mathbf{x}|} \left(1 + i\frac{\epsilon}{|\mathbf{k}|} \right)
\end{aligned}$$

Taking the limit $\epsilon \to 0$, we recover the Green's function for the Helmholtz operator in position space corresponding to "outgoing" waves

$$\boxed{G^+_{\mathbf{k}}(\mathbf{x}) = -\frac{1}{4\pi} \frac{e^{i|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|}}$$

Had we considered considered incoming waves, all that would change is the replacement $|\mathbf{k}| \to -|\mathbf{k}|$. This would cause $\epsilon \to -\epsilon$ in the denominator of the momentum-space Green's function, and consequently the locations of the poles would switch in terms of which is above and which is below the real q axis. This would ultimately change which residue is calculated and return a position-space Green's function with a negative exponent, perfectly agreeing with the initial change to the direction of the momentum.