Chapter 9

9.1

The translation operator is given by

$$\hat{U}(\mathbf{a}) = e^{-i\hat{\mathbf{p}}\cdot\mathbf{a}}$$

In this simple form, the generator of this transformation, $\hat{\mathbf{p}}$, can be expressed as the following derivative

$$\boxed{\hat{\mathbf{p}} = i\frac{\partial \hat{U}}{\partial \mathbf{a}}\bigg|_{\mathbf{a}=0}}$$

9.2

For the representations of Lorentz boosts shown below

$$D(\phi^1) = \begin{pmatrix} \cosh\phi^1 & \sinh\phi^1 & 0 & 0 \\ \sinh\phi^1 & \cosh\phi^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D(\phi^2) = \begin{pmatrix} \cosh\phi^2 & 0 & \sinh\phi^2 & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi^2 & 0 & \cosh\phi^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D(\phi^3) = \begin{pmatrix} \cosh\phi^3 & 0 & 0 & \sinh\phi^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi^3 & 0 & 0 & \cosh\phi^3 \end{pmatrix}$$

The generators of these transformations can be represented as follows

$$\boxed{K^1 = -i\frac{\partial D(\phi^1)}{\partial \phi^1}\bigg|_{\phi^1=0} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\boxed{K^2 = -i\frac{\partial D(\phi^2)}{\partial \phi^2}\bigg|_{\phi^2=0} = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$\boxed{K^3 = -i\frac{\partial D(\phi^3)}{\partial \phi^3}\bigg|_{\phi^3=0} = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$$

9.3

A infinitesimal boost ϕ^j along the x^j axis can be expressed as successive boosts along the three cardinal axes. For infinitesimal boosts we have

$$\cosh\phi \approx 1, \quad \sinh\phi \approx \phi$$

Therefore, we have

$$\begin{aligned}
\Lambda(\phi^j)^\mu{}_\nu &= \Lambda(\phi^1)^\mu{}_\alpha \Lambda(\phi^2)^\alpha{}_\beta \Lambda(\phi^3)^\beta{}_\nu \\
&= \begin{pmatrix} 1 & \phi^1 & 0 & 0 \\ \phi^1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \phi^2 & 0 \\ 0 & 1 & 0 & 0 \\ \phi^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \phi^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \phi^3 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\boxed{\Lambda(\phi^j)^\mu{}_\nu = \begin{pmatrix} 1 & \phi^1 & \phi^2 & \phi^3 \\ \phi^1 & 1 & 0 & 0 \\ \phi^2 & 0 & 1 & 0 \\ \phi^3 & 0 & 0 & 1 \end{pmatrix}}$$

Rotations can be represented as

$$R(\theta^1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta^1 & \sin\theta^1 \\ 0 & 0 & -\sin\theta^1 & \cos\theta^1 \end{pmatrix}, \quad R(\theta^2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta^2 & 0 & -\sin\theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta^2 & 0 & \cos\theta^2 \end{pmatrix}$$

$$R(\theta^3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta^3 & \sin\theta^3 & 0 \\ 0 & -\sin\theta^3 & \cos\theta^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A infinitesimal rotation θ^j about the x^j axis can be expressed as successive rotations about the three cardinal axes. For infinitesimal rotations we have

$$\cos\theta \approx 1, \quad \sin\theta \approx \theta$$

Therefore, we have

$$\begin{aligned}
\Lambda(\theta^j)^\mu{}_\nu &= \Lambda(\theta^1)^\mu{}_\alpha \Lambda(\theta^2)^\alpha{}_\beta \Lambda(\theta^3)^\beta{}_\nu \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta^1 \\ 0 & 0 & -\theta^1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta^2 \\ 0 & 0 & 1 & 0 \\ 0 & \theta^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & 0 \\ 0 & -\theta^3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\boxed{\Lambda(\theta^j)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & -\theta^2 \\ 0 & -\theta^3 & 1 & \theta^1 \\ 0 & \theta^2 & -\theta^1 & 1 \end{pmatrix}}$$

Using this, a general infinitesimal Lorentz transformation can be written as

$$\begin{aligned}
\mathbf{\Lambda} &= \Lambda(\phi^i)^\mu{}_\alpha \Lambda(\theta^j)^\alpha{}_\nu \\
&= \begin{pmatrix} 1 & \phi^1 & \phi^2 & \phi^3 \\ \phi^1 & 1 & \theta^3 & -\theta^2 \\ \phi^2 & -\theta^3 & 1 & \theta^1 \\ \phi^3 & \theta^2 & -\theta^1 & 1 \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{\Lambda} = \mathbb{1} + \boldsymbol{\omega}}$$

where

$$\boldsymbol{\omega} = \omega^\mu{}_\nu = \begin{pmatrix} 0 & \phi^1 & \phi^2 & \phi^3 \\ \phi^1 & 0 & \theta^3 & -\theta^2 \\ \phi^2 & -\theta^3 & 0 & \theta^1 \\ \phi^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}$$

This tensor, when made doubly contravariant or covariant, is anti-symmetric

$$\omega^{\mu\nu} = \omega^{\mu}{}_{\lambda} g^{\lambda\nu}$$
$$= \begin{pmatrix} 0 & \phi^1 & \phi^2 & \phi^3 \\ \phi^1 & 0 & \theta^3 & -\theta^2 \\ \phi^2 & -\theta^3 & 0 & \theta^1 \\ \phi^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$\boxed{\omega^{\mu\nu} = \begin{pmatrix} 0 & -\phi^1 & -\phi^2 & -\phi^3 \\ \phi^1 & 0 & -\theta^3 & \theta^2 \\ \phi^2 & \theta^3 & 0 & -\theta^1 \\ \phi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}}$$

$$\omega_{\mu\nu} = g_{\mu\lambda} \omega^{\lambda}{}_{\nu}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \phi^1 & \phi^2 & \phi^3 \\ \phi^1 & 0 & \theta^3 & -\theta^2 \\ \phi^2 & -\theta^3 & 0 & \theta^1 \\ \phi^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}$$
$$\boxed{\omega_{\mu\nu} = \begin{pmatrix} 0 & \phi^1 & \phi^2 & \phi^3 \\ -\phi^1 & 0 & \theta^3 & -\theta^2 \\ -\phi^2 & -\theta^3 & 0 & \theta^1 \\ -\phi^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}}$$

We can write an explicit relationship between ϕ^i, θ^i and ω^{ij} as follows

$$\boxed{\phi^i = \omega^{0i}}$$

which is easily verified just by looking at $\omega^{\mu\nu}$. As for θ^i, we have

$$\boxed{\theta^i = -\frac{1}{2} \varepsilon^{ijk} \omega^{jk}}$$

where summation is implied over j, k. This we can check by hand

$$\begin{aligned} \theta^1 &= -\frac{1}{2}\left(\omega^{23} - \omega^{32}\right) & \theta^2 &= -\frac{1}{2}\left(\omega^{31} - \omega^{13}\right) & \theta^3 &= -\frac{1}{2}\left(\omega^{12} - \omega^{21}\right) \\ &= -\omega^{23} & &= \omega^{31} & &= -\omega^{12} \\ \theta^1 &= \theta^1 & \theta^2 &= \theta^2 & \theta^3 &= \theta^3 \end{aligned}$$

9.4

Under a Poincaré transformation, a function will transform as

$$f(x') = f(x) + a^\mu \partial_\mu f(x) + \omega^{\mu}{}_{\nu} x^\nu \partial_\mu f(x)$$

Since $\omega_{\mu\nu} = -\omega_{\nu\mu}$, we can write this as

$$\begin{aligned} f(x') &= \left[1 + a^\mu \partial_\mu + g^{\mu\lambda} \omega_{\lambda\nu} x^\nu \partial_\mu\right] f(x) \\ &= \left[1 + a^\mu \partial_\mu + \omega_{\lambda\nu} x^\nu \partial^\lambda\right] f(x) \\ &= \left[1 + a^\mu \partial_\mu + \frac{1}{2}(\omega_{\lambda\nu} - \omega_{\nu\lambda}) x^\nu \partial^\lambda\right] f(x) \\ &= \left[1 + a^\mu \partial_\mu + \frac{1}{2}\omega_{\lambda\nu}\left(x^\nu \partial^\lambda - x^\lambda \partial^\nu\right)\right] f(x) \end{aligned}$$
$$\boxed{f(x') = \left[1 + a^\mu \partial_\mu - \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu)\right] f(x)}$$

If we define the generators of the Poincaré group as $p_\mu = -i\partial_\mu$ and $M^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu)$, then we can write this as

$$\boxed{f(x') = \left[1 - ia^\mu p_\mu + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right]f(x)}$$

Since $M^{\mu\nu}$ generates Lorentz transformation, it can be related to the generators of rotations and boosts as

$$J^i = \frac{1}{2}\varepsilon^{ijk}M^{jk}, \quad K^i = M^{0i}$$

Putting these together with how $\omega^{\mu\nu}$ relations to rotations and boosts, we can write

$$\begin{aligned}
\omega_{\mu\nu}M^{\mu\nu} &= 2\omega_{0\mu}M^{0\mu} + 2\omega_{i\mu}M^{i\mu} \\
&= 2\omega_{0i}M^{0i} + 2\omega_{ij}M^{ij} \\
\omega_{\mu\nu}M^{\mu\nu} &= -2\phi_i K^i + 2\omega_{ij}M^{ij}
\end{aligned}$$

Putting together θ_i and J^i, we have

$$\begin{aligned}
\theta_i J^i &= \frac{1}{4}\varepsilon^{ijk}\varepsilon_{imn}\omega_{jk}M^{mn} \\
&= \frac{1}{4}\left(\delta^j_m\delta^k_n - \delta^j_n\delta^k_m\right)\omega_{jk}M^{mn} \\
&= \frac{1}{2}\omega_{jk}\left(M^{jk} - M^{kj}\right) \\
\theta_i J^i &= \omega_{jk}M^{jk}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\omega_{\mu\nu}M^{\mu\nu} &= -2\phi_i K^i + 2\theta_i J^i \\
-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} &= -i(J^i\theta_i - K^i\phi_i)
\end{aligned}$$

$$\boxed{\exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) = e^{-i(\mathbf{J}\cdot\boldsymbol{\theta} - \mathbf{K}\cdot\boldsymbol{\phi})}}$$