

Chapter 14

14.1

For the electromagnetic Lagrangian density in vacuum

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$$

the conjugate momentum to A^μ is

$$\pi^0 = 0, \quad \pi^i = \partial^i A^0 - \partial^0 A^i$$

Making the identification $E^i = -F^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$, we can write the Hamiltonian in terms of the electric and magnetic fields as

$$\mathcal{H} = \frac{1}{2}(\mathbf{E} + \mathbf{B})^2 - A^0(\nabla \cdot \mathbf{E})$$

or in terms of the conjugate variables as

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{\pi}^2 - \mathbf{A} \cdot \nabla^2 \mathbf{A} + (\mathbf{A} \cdot \nabla)(\nabla \cdot \mathbf{A})) - A^0(\nabla \cdot \boldsymbol{\pi})$$

If we work in the Coulomb gauge, then $\nabla \cdot \mathbf{A} = 0$, and it then follows from the equations of motion that $A^0 = 0$ (but only in vacuum). Therefore, we can write

$$\mathcal{H} = \frac{1}{2}(\pi_\mu \pi^\mu - A_\mu \nabla^2 A^\mu)$$

With the decomposition

$$\begin{aligned} A^\mu(x) &= \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \sum_\lambda \left(\epsilon_\lambda^\mu(p) a_{\mathbf{p}\lambda} e^{-ip \cdot x} + \epsilon_\lambda^{\mu*}(p) a_{\mathbf{p}\lambda}^\dagger e^{ip \cdot x} \right) = -A_\mu(x) \\ \pi^\mu(x) &= i \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{E_p}{\sqrt{2E_p}} \sum_\lambda \left(\epsilon_\lambda^\mu(p) a_{\mathbf{p}\lambda} e^{-ip \cdot x} - \epsilon_\lambda^{\mu*}(p) a_{\mathbf{p}\lambda}^\dagger e^{ip \cdot x} \right) = -\pi_\mu(x) \\ \nabla^2 A^\mu(x) &= - \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{|\mathbf{p}|^2}{\sqrt{2E_p}} \sum_\lambda \left(\epsilon_\lambda^\mu(p) a_{\mathbf{p}\lambda} e^{-ip \cdot x} + \epsilon_\lambda^{\mu*}(p) a_{\mathbf{p}\lambda}^\dagger e^{ip \cdot x} \right) \end{aligned}$$

the Hamiltonian becomes

$$\begin{aligned} H &= \frac{1}{2} \int d^3 x (\pi_\mu \pi^\mu - A_\mu \nabla^2 A^\mu) \\ &= \frac{1}{2} \int \frac{d^3 x \, d^3 p \, d^3 q}{(2\pi)^3 2 \sqrt{E_p E_q}} \sum_{\lambda \lambda'} \left[E_p E_q \left(\epsilon_{\mu\lambda} a_{\mathbf{p}\lambda} e^{-ip \cdot x} - \epsilon_{\mu\lambda}^* a_{\mathbf{p}\lambda}^\dagger e^{ip \cdot x} \right) \left(\epsilon_{\lambda' \lambda'}^\mu a_{\mathbf{q}\lambda'} e^{-iq \cdot x} - \epsilon_{\lambda' \lambda'}^{\mu*} a_{\mathbf{q}\lambda'}^\dagger e^{iq \cdot x} \right) \right. \\ &\quad \left. - |\mathbf{q}|^2 \left(\epsilon_{\mu\lambda} a_{\mathbf{p}\lambda} e^{-ip \cdot x} + \epsilon_{\mu\lambda}^* a_{\mathbf{p}\lambda}^\dagger e^{ip \cdot x} \right) \left(\epsilon_{\lambda' \lambda'}^\mu a_{\mathbf{q}\lambda'} e^{-iq \cdot x} + \epsilon_{\lambda' \lambda'}^{\mu*} a_{\mathbf{q}\lambda'}^\dagger e^{iq \cdot x} \right) \right] \\ &= \frac{1}{2} \int \frac{d^3 x \, d^3 p \, d^3 q}{(2\pi)^3 2 \sqrt{E_p E_q}} \sum_{\lambda \lambda'} \left[\left(E_p E_q - |\mathbf{q}|^2 \right) \left(\epsilon_{\mu\lambda} \epsilon_{\lambda'}^\mu a_{\mathbf{p}\lambda} a_{\mathbf{q}\lambda'} e^{-i(p+q) \cdot x} + \epsilon_{\mu\lambda}^* \epsilon_{\lambda'}^{\mu*} a_{\mathbf{p}\lambda}^\dagger a_{\mathbf{q}\lambda'}^\dagger e^{i(p+q) \cdot x} \right) \right. \\ &\quad \left. - \left(E_p E_q + |\mathbf{q}|^2 \right) \left(\epsilon_{\mu\lambda} \epsilon_{\lambda'}^{\mu*} a_{\mathbf{p}\lambda}^\dagger a_{\mathbf{q}\lambda'} e^{-i(p-q) \cdot x} + \epsilon_{\mu\lambda}^* \epsilon_{\lambda'}^\mu a_{\mathbf{p}\lambda} a_{\mathbf{q}\lambda'}^\dagger e^{i(p-q) \cdot x} \right) \right] \end{aligned}$$

$$\begin{aligned}
H &= \frac{1}{2} \int \frac{d^3 p d^3 q}{2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}} \sum_{\lambda \lambda'} \left[\left(E_{\mathbf{p}} E_{\mathbf{q}} - |\mathbf{q}|^2 \right) \left(\epsilon_{\mu \lambda} \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda} a_{\mathbf{q} \lambda'} e^{-i(E_{\mathbf{p}} + E_{\mathbf{q}})t} + \right. \right. \\
&\quad + \epsilon_{\mu \lambda}^* \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{q} \lambda'}^\dagger e^{i(E_{\mathbf{p}} + E_{\mathbf{q}})t} \Big) \delta^{(3)}(\mathbf{p} + \mathbf{q}) - \left(E_{\mathbf{p}} E_{\mathbf{q}} + |\mathbf{q}|^2 \right) \left(\epsilon_{\mu \lambda} \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda} a_{\mathbf{q} \lambda'}^\dagger e^{-i(E_{\mathbf{p}} - E_{\mathbf{q}})t} + \right. \\
&\quad \left. \left. + \epsilon_{\mu \lambda}^* \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{q} \lambda'} e^{i(E_{\mathbf{p}} - E_{\mathbf{q}})t} \right) \delta^{(3)}(\mathbf{p} - \mathbf{q}) \right] \\
&= \frac{1}{2} \int \frac{d^3 p}{2E_{\mathbf{p}}} \sum_{\lambda \lambda'} \left[\left(E_{\mathbf{p}}^2 - |\mathbf{p}|^2 \right) \left(\epsilon_{\mu \lambda} \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda} a_{-\mathbf{p} \lambda'} e^{-2iE_{\mathbf{p}} t} + \epsilon_{\mu \lambda}^* \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda}^\dagger a_{-\mathbf{p} \lambda'}^\dagger e^{2iE_{\mathbf{p}} t} \right) \right. \\
&\quad \left. - \left(E_{\mathbf{p}}^2 + |\mathbf{p}|^2 \right) \left(\epsilon_{\mu \lambda} \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda} a_{\mathbf{p} \lambda'}^\dagger + \epsilon_{\mu \lambda}^* \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{p} \lambda'} \right) \right] \\
&= -\frac{1}{2} \int d^3 p E_{\mathbf{p}} \sum_{\lambda \lambda'} \left(\epsilon_{\mu \lambda} \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda} a_{\mathbf{p} \lambda'}^\dagger + \epsilon_{\mu \lambda}^* \epsilon_{\lambda'}^\mu a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{p} \lambda'} \right) \\
&= \frac{1}{2} \int d^3 p E_{\mathbf{p}} \sum_{\lambda \lambda'} \left(a_{\mathbf{p} \lambda} a_{\mathbf{p} \lambda'}^\dagger + a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{p} \lambda'} \right) \delta_{\lambda \lambda'} \\
H &= \frac{1}{2} \int d^3 p E_{\mathbf{p}} \sum_{\lambda} \left(a_{\mathbf{p} \lambda} a_{\mathbf{p} \lambda}^\dagger + a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{p} \lambda} \right)
\end{aligned}$$

Normal ordering, we obtain the quantized Hamiltonian for the electromagnetic field in vacuum

$$\boxed{H = \int d^3 p \sum_{\lambda} E_{\mathbf{p}} a_{\mathbf{p} \lambda}^\dagger a_{\mathbf{p} \lambda}}$$

14.2

For a photon propagating in the z-direction $q^\mu = (|\mathbf{q}| \ 0 \ 0 \ |\mathbf{q}|)^T$, we can choose polarization vectors

$$\epsilon_1^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_2^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The operator S^z has commutation relations with the photon creation and annihilation operators

$$[S^z, a_{\mathbf{q} \lambda}^\dagger] = i\epsilon_1^{1*} a_{\mathbf{q} 2}^\dagger - i\epsilon_2^{2*} a_{\mathbf{q} 1}^\dagger$$

These creation operators create photons in states of linear polarization. Defining creation operators for circular polarization

$$b_{\mathbf{q} R}^\dagger = -\frac{1}{\sqrt{2}} \left(a_{\mathbf{q} 1}^\dagger + ia_{\mathbf{q} 2}^\dagger \right), \quad b_{\mathbf{q} L}^\dagger = \frac{1}{\sqrt{2}} \left(a_{\mathbf{q} 1}^\dagger - ia_{\mathbf{q} 2}^\dagger \right)$$

we find the commutation relations

$$\begin{aligned}
[S^z, b_{\mathbf{q} R}^\dagger] &= -\frac{1}{\sqrt{2}} \left([S^z, a_{\mathbf{q} 1}^\dagger] + i[S^z, a_{\mathbf{q} 2}^\dagger] \right) \\
&= -\frac{1}{\sqrt{2}} \left(i\epsilon_1^{1*} a_{\mathbf{q} 2}^\dagger - i\epsilon_1^{2*} a_{\mathbf{q} 1}^\dagger - \epsilon_2^{1*} a_{\mathbf{q} 2}^\dagger + \epsilon_2^{2*} a_{\mathbf{q} 1}^\dagger \right) \\
&= -\frac{1}{\sqrt{2}} \left(a_{\mathbf{q} 1}^\dagger + ia_{\mathbf{q} 2}^\dagger \right)
\end{aligned}$$

$$\boxed{[S^z, b_{\mathbf{q} R}^\dagger] = b_{\mathbf{q} R}^\dagger}$$

$$\begin{aligned}
[S^z, b_{\mathbf{q}L}^\dagger] &= \frac{1}{\sqrt{2}} \left([S^z, a_{\mathbf{q}1}^\dagger] - i[S^z, a_{\mathbf{q}2}^\dagger] \right) \\
&= \frac{1}{\sqrt{2}} \left(i\epsilon_1^{1*} a_{\mathbf{q}2}^\dagger - i\epsilon_1^{2*} a_{\mathbf{q}1}^\dagger + \epsilon_2^{1*} a_{\mathbf{q}2}^\dagger - \epsilon_2^{2*} a_{\mathbf{q}1}^\dagger \right) \\
&= -\frac{1}{\sqrt{2}} \left(a_{\mathbf{q}1}^\dagger - ia_{\mathbf{q}2}^\dagger \right) \\
\boxed{[S^z, b_{\mathbf{q}L}^\dagger] = -b_{\mathbf{q}L}^\dagger}
\end{aligned}$$

With this, we can consider the action of the spin operator on a one-photon state with circular polarization

$$\begin{aligned}
S^z |\gamma_\lambda\rangle &= S^z b_{\mathbf{q}\lambda}^\dagger |0\rangle \\
&= \left(b_{\mathbf{q}\lambda}^\dagger S^z + [S^z, b_{\mathbf{q}\lambda}^\dagger] \right) |0\rangle \\
S^z |\gamma_\lambda\rangle &= b_{\mathbf{q}\lambda} S^z |0\rangle + \zeta(\lambda) b_{\mathbf{q}\lambda}^\dagger |0\rangle
\end{aligned}$$

where $\zeta(\lambda) = \pm 1$ for $\lambda = R, L$ respectively. Since there are no particles in the vacuum, $S^z |0\rangle = 0$ and we have

$$\boxed{S^z |\gamma_\lambda\rangle = \pm |\gamma_\lambda\rangle, \quad \lambda = R, L}$$

Thus, the photon can either have spin projection +1 or -1, but not zero.