Chapter 13

13.1

The conserved charges for an SO(3) internal symmetry

$$\hat{Q}^a_{\mathrm{N}} = -i \int \mathrm{d}^3 p \, \varepsilon^{abc} \hat{a}^\dagger_{b\mathbf{p}} \hat{a}_{c\mathbf{p}}$$

can be written collectively as

$$\hat{\mathbf{Q}}_{\mathrm{N}} = \int \mathrm{d}^3 p \, \hat{\mathbf{A}}^\dagger_{\mathbf{p}} \mathbf{J} \hat{\mathbf{A}}_{\mathbf{p}}$$

where $\hat{\mathbf{A}}_{\mathbf{p}} = \begin{pmatrix} \hat{a}_{1\mathbf{p}} & \hat{a}_{2\mathbf{p}} & \hat{a}_{3\mathbf{p}} \end{pmatrix}^{\mathrm{T}}$ and the vector of matrices $\mathbf{J}$ must be of the form such that we recover the component-wise formulae given above. These matrices turn out to be precisely the spin 1 angular momentum matrices

$$J_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which can be readily checked by doing out the matrix multiplication. Therefore, the Noether charge is given by

$$\boxed{\hat{\mathbf{Q}}_{\mathrm{N}} = \int \mathrm{d}^3 p \, \hat{\mathbf{A}}^\dagger_{\mathbf{p}} \mathbf{J} \hat{\mathbf{A}}_{\mathbf{p}}}$$

To write the Noether charge in the form

$$\hat{\mathbf{Q}}_{\mathrm{N}} = \int \mathrm{d}^3 p \, \hat{\mathbf{B}}^\dagger_{\mathbf{p}} \tilde{\mathbf{J}} \hat{\mathbf{B}}_{\mathbf{p}}$$

where $\hat{\mathbf{B}}_{\mathbf{p}} = \begin{pmatrix} -\frac{\hat{a}_{1\mathbf{p}} - i\hat{a}_{2\mathbf{p}}}{\sqrt{2}} & \hat{a}_{3\mathbf{p}} & \frac{\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}}}{\sqrt{2}} \end{pmatrix}$, we need the unitary matrix U such that $\hat{\mathbf{B}}_{\mathbf{p}} = U \hat{\mathbf{A}}_{\mathbf{p}}$. This matrix has the form

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix}$$

With this, we have $\tilde{\mathbf{J}} = U \mathbf{J} U^\dagger$ from which we can read off

$$\boxed{\tilde{J}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{J}_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{J}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

13.2

To confirm that the matrix

$$\Lambda^{\mu}{}_{\nu}(p) = \frac{1}{m}\begin{pmatrix} E_{\mathbf{p}} & 0 & 0 & |\mathbf{p}| \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ |\mathbf{p}| & 0 & 0 & E_{\mathbf{p}} \end{pmatrix}$$

performs a boost in the z-direction, we can simply apply the matrix to the four-vector of a massive particle at rest

$$\boxed{\Lambda^{\mu}{}_{\nu}(p)\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_{\mathbf{p}} \\ 0 \\ 0 \\ |\mathbf{p}| \end{pmatrix}}$$

which yields the four-vector of a massive particle moving with momentum $|\mathbf{p}|$ in the z-direction. After boosting the polarization vectors, we can check that they remain normalized according to

$$\epsilon^{*}_{\lambda} \cdot \epsilon_{\lambda} = g_{\mu\nu}\epsilon^{\mu *}_{\lambda}(p)\epsilon^{\nu}_{\lambda}(p) = -1$$

as follows

$$\boxed{\epsilon^{*}_{1} \cdot \epsilon_{1} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = -1}$$

$$\boxed{\epsilon^{*}_{2} \cdot \epsilon_{2} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -1}$$

$$\boxed{\epsilon^{*}_{3} \cdot \epsilon_{3} = \begin{pmatrix} |\mathbf{p}|/m & 0 & 0 & E_{\mathbf{p}}/m \end{pmatrix}\begin{pmatrix} |\mathbf{p}|/m \\ 0 \\ 0 \\ -E_{\mathbf{p}}/m \end{pmatrix} = \frac{|\mathbf{p}|^2 - E_{\mathbf{p}}^2}{m^2} = -1}$$

If we instead consider circular polarization vectors

$$\epsilon^{\mu}_{\lambda=R} = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad \epsilon^{\mu}_{\lambda=L} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \epsilon^{\mu}_{\lambda=3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we find that

$$\boxed{\epsilon^{*}_{R} \cdot \epsilon_{R} = \frac{1}{2}\begin{pmatrix} 0 & 1 & i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix} = -1}$$

$$\boxed{\epsilon^{*}_{L} \cdot \epsilon_{L} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} = -1}$$

$$\boxed{\epsilon^{*}_{3} \cdot \epsilon_{3} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = -1}$$

these polarization vectors are normalized by the same convention.

13.3

To check that $P_L^{\mu\nu}$ and $P_T^{\mu\nu}$ are indeed projection operators, we need to confirm that ${P_L}^2 = P_L$ and ${P_T}^2 = P_T$. Starting with the former, we have

$$\begin{aligned}
(P_L^{\mu\nu})^2 &= P_L^{\mu\sigma} g_{\sigma\lambda} P_L^{\lambda\nu} \\
&= \left(\frac{p^\mu p_\lambda}{p^2}\right)\left(\frac{p^\lambda p^\nu}{p^2}\right) \\
&= \frac{p^\mu p^\nu}{p^2} \\
\boxed{(P_L^{\mu\nu})^2 = P_L^{\mu\nu}}
\end{aligned}$$

The latter similarly yields

$$\begin{aligned}
(P_T^{\mu\nu})^2 &= P_T^{\mu\sigma} g_{\sigma\lambda} P_T^{\lambda\nu} \\
&= \left({g^\mu}_\lambda - \frac{p^\mu p_\lambda}{p^2}\right)\left(g^{\lambda\nu} - \frac{p^\lambda p^\nu}{p^2}\right) \\
&= {\delta^\mu}_\lambda g^{\lambda\nu} - {\delta^\mu}_\lambda \frac{p^\lambda p^\nu}{p^2} - g^{\lambda\nu}\frac{p^\mu p_\lambda}{p^2} + \frac{p^\mu p^\nu}{p^2} \\
&= g^{\mu\nu} - 2\frac{p^\mu p^\nu}{p^2} + \frac{p^\mu p^\nu}{p^2} \\
&= g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \\
\boxed{(P_T^{\mu\nu})^2 = P_T^{\mu\nu}}
\end{aligned}$$

13.4

Starting with the electromagnetic Lagrangian in vacuo

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

we can plug in the definition of the field strength tensor to write this as

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\boxed{\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)}$$

Doing an integration by parts on each term (with the understanding that what we're really considering is the action, which is the integral of $\mathcal{L}$) and enforcing that the field A_μ vanish on the boundary, this becomes

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(A_\nu \partial_\mu \partial^\mu A^\nu - A_\nu \partial_\mu \partial^\nu A^\mu) \\
&= \frac{1}{2}\big(A^\mu g_{\mu\nu} \partial^2 A^\nu - A^\mu \partial_\mu \partial_\nu A^\nu\big) \\
\mathcal{L} &= \frac{1}{2} A^\mu \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right)\partial^2 A^\nu
\end{aligned}$$

Making the identification that $p_\mu = i\partial_\mu$, this becomes

$$\mathcal{L} = \frac{1}{2} A^\mu \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\partial^2 A^\nu$$

which we recognize as containing the aforementioned projection operator

$$\boxed{\mathcal{L} = \frac{1}{2} A^\mu P^T_{\mu\nu} \partial^2 A^\nu}$$