

Chapter 8

8.1

The properties of the time evolution can be demonstrated below

$$\boxed{\hat{U}(t_1, t_1) = e^{-i\hat{H}(t_1 - t_1)} = \mathbb{1}}$$

$$\begin{aligned}\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) &= e^{-i\hat{H}(t_3 - t_2)}e^{-i\hat{H}(t_2 - t_1)} \\ &= e^{-i\hat{H}(t_3 - t_1)}\end{aligned}$$

$$\boxed{\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = \hat{U}(t_3, t_1)}$$

$$\begin{aligned}i\frac{d}{dt_2}\hat{U}(t_2, t_1) &= i\frac{d}{dt_2}e^{-i\hat{H}(t_2 - t_1)} \\ i\frac{d}{dt_2}\hat{U}(t_2, t_1) &= \hat{H}\hat{U}(t_2, t_1)\end{aligned}$$

$$\boxed{i\frac{d}{dt_2}\hat{U}(t_2, t_1) = \hat{H}\hat{U}(t_2, t_1)}$$

$$\begin{aligned}\hat{U}(t_2, t_1)\hat{U}(t_1, t_2) &= e^{-i\hat{H}(t_2 - t_1)}e^{-i\hat{H}(t_1 - t_2)} \\ &= \mathbb{1}\end{aligned}$$

$$\boxed{\hat{U}(t_1, t_2) = \hat{U}^{-1}(t_2, t_1)}$$

$$\begin{aligned}\hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) &= e^{i\hat{H}(t_2 - t_1)}e^{-i\hat{H}(t_2 - t_1)} \\ \hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) &= \mathbb{1}\end{aligned}$$

$$\boxed{\hat{U}^\dagger(t_2, t_1)\hat{U}(t_2, t_1) = \mathbb{1}}$$

8.2

For the Hamiltonian

$$\hat{H} = \sum_k E_k \hat{a}_k^\dagger \hat{a}_k$$

we can use the Heisenberg equations of motion to write

$$\begin{aligned}
i\hbar \frac{d\hat{a}_k^\dagger}{dt} &= [\hat{a}_k^\dagger(t), \hat{H}] \\
&= [\hat{U}^\dagger(t)\hat{a}_k^\dagger\hat{U}(t), \hat{H}] \\
&= [e^{i\hat{H}t/\hbar}\hat{a}_k^\dagger e^{-i\hat{H}t/\hbar}, \hat{H}] \\
&= e^{i\hat{H}t/\hbar}[\hat{a}_k^\dagger, \hat{H}]e^{-i\hat{H}t/\hbar} \\
&= e^{i\hat{H}t/\hbar} \sum_\ell E_\ell [\hat{a}_k^\dagger, \hat{a}_\ell^\dagger \hat{a}_\ell] e^{-i\hat{H}t/\hbar} \\
&= -e^{i\hat{H}t/\hbar} \sum_\ell E_\ell \hat{a}_\ell^\dagger \delta_{k,\ell} e^{-i\hat{H}t/\hbar} \\
&= -E_k e^{i\hat{H}t/\hbar} \hat{a}_k^\dagger e^{-i\hat{H}t/\hbar} \\
i\hbar \frac{d\hat{a}_k}{dt} &= -E_k \hat{a}_k^\dagger(t)
\end{aligned}$$

which gives as a function of time

$$\boxed{\hat{a}_k^\dagger(t) = \hat{a}_k^\dagger e^{iE_k t/\hbar}}$$

This process can be repeated for $\hat{a}_k(t)$ with the only change being a sign from the commutator $[\hat{a}, \hat{n}]$, yielding

$$\boxed{\hat{a}_k(t) = \hat{a}_k e^{-iE_k t/\hbar}}$$

8.3

For the operator

$$\hat{X} = X_{\ell m} \hat{a}_\ell^\dagger \hat{a}_m$$

the time evolution is given by

$$\begin{aligned}
\hat{X}(t) &= e^{i\hat{H}t/\hbar} \hat{X} e^{-i\hat{H}t/\hbar} \\
&= X_{\ell m} e^{i\hat{H}t/\hbar} \hat{a}_\ell^\dagger \mathbb{1} \hat{a}_m e^{-i\hat{H}t/\hbar} \\
&= X_{\ell m} \left(e^{i\hat{H}t/\hbar} \hat{a}_\ell^\dagger e^{-i\hat{H}t/\hbar} \right) \left(e^{i\hat{H}t/\hbar} \hat{a}_m e^{-i\hat{H}t/\hbar} \right) \\
&= X_{\ell m} \hat{a}_\ell^\dagger(t) \hat{a}_m(t) \\
&= X_{\ell m} \hat{a}_\ell^\dagger e^{iE_\ell t/\hbar} \hat{a}_m e^{-iE_m t/\hbar} \\
&= X_{\ell m} \hat{a}_\ell^\dagger \hat{a}_m e^{i(E_\ell - E_m)t/\hbar} \\
\boxed{\hat{X}(t) = \hat{X} e^{i(E_\ell - E_m)t/\hbar}}
\end{aligned}$$

8.4

For the Hamiltonian

$$\hat{H} = \omega \hat{S}_y$$

we can use the Hamiltonian equations of motion to write

$$\begin{aligned}
i\hbar \frac{d\hat{S}_z}{dt} &= [\hat{S}_z, \hat{H}] \\
&= \omega [\hat{S}_z, \hat{S}_y] \\
\boxed{\frac{d\hat{S}_z}{dt} = -\omega \hat{S}_x}
\end{aligned}$$

as well as

$$\begin{aligned} i\hbar \frac{d\hat{S}_x}{dt} &= [\hat{S}_x, \hat{H}] \\ &= \omega [\hat{S}_x, \hat{S}_y] \\ \boxed{\frac{d\hat{S}_x}{dt} = \omega \hat{S}_z} \end{aligned}$$

These coupled equations of motion can be decoupled as follows

$$\begin{aligned} \frac{d^2 \hat{S}_z}{dt^2} &= -\omega \frac{d\hat{S}_x}{dt} = -\omega^2 \hat{S}_z \\ \frac{d^2 \hat{S}_x}{dt^2} &= \omega \frac{d\hat{S}_z}{dt} = -\omega^2 \hat{S}_x \end{aligned}$$

which have solutions

$$\begin{aligned} \hat{S}_x(t) &= \hat{S}_z(0) \sin \omega t \\ \hat{S}_z(t) &= \hat{S}_x(0) \cos \omega t \end{aligned}$$

This shows that the spin precesses about the external magnetic field.