Chapter 7

7.1

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \sum_{n=1}^{\infty} \lambda_n \varphi^{2n+2}$$

the equations of motion are given by

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$$

$$-m^2\varphi^2 - \sum_{n=1}^{\infty} \lambda_n (2n+2)\varphi^{2n+1} - \partial_\mu \partial^\mu \varphi = 0$$

$$\boxed{(\partial^2 + m^2)\varphi + \sum_{n=1}^{\infty} (2n+2)\lambda_n \varphi^{2n+1} = 0}$$

7.2

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 + J(x)\varphi$$

the equations of motion are given by

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$$

$$-m^2\varphi + J(x) - \partial_\mu \partial^\mu \varphi = 0$$

$$\boxed{(\partial^2 + m^2)\varphi = J(x)}$$

7.3

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_1)^2 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - \frac{1}{2}m^2\varphi_2^2 - g(\varphi_1^2 + \varphi_2^2)^2$$

the equations of motion are given by

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_1)} \right) = 0$$

$$-m^2\varphi_1 - 4g\varphi_1(\varphi_1^2 + \varphi_2^2) - \partial_\mu \partial^\mu \varphi_1 = 0$$

$$\boxed{(\partial^2 + m^2)\varphi_1 + 4g(\varphi_1^2 + \varphi_2^2)^2\varphi_1 = 0}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_2} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_2)} \right) = 0$$
$$-m^2\varphi_2 - 4g\varphi_2(\varphi_1^2 + \varphi_2^2) - \partial_\mu \partial^\mu \varphi_2 = 0$$
$$\boxed{(\partial^2 + m^2)\varphi_2 + 4g(\varphi_1^2 + \varphi_2^2)^2\varphi_2 = 0}$$

7.4

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2$$

we can write out the summation as

$$\boxed{\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 - \frac{1}{2}m^2\varphi^2}$$

In this form, it's clear to see that

$$\boxed{\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}}$$

and that

$$\begin{aligned}\mathcal{H} &= \pi\dot{\varphi} - \mathcal{L} \\ &= \dot{\varphi}^2 - \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2\end{aligned}$$
$$\boxed{\mathcal{H} = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2}$$

If we define

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}$$

then for this Lagrangian we have

$$\boxed{\Pi^\mu = \partial^\mu \varphi, \quad \Pi^0 = \pi = \dot{\varphi}}$$