Chapter 5

5.1

For a Lagrangian that depends explicitly on time, we have

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial x_i}\dot{x}_i + \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i + \frac{\partial L}{\partial t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\dot{x}_i + \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i + \frac{\partial L}{\partial t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i\right) + \frac{\partial L}{\partial t} \\
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i\dot{x}_i\right) + \frac{\partial L}{\partial t} \\
\frac{\partial L}{\partial t} &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i\dot{x}_i - L\right) \\
&\boxed{\frac{\partial L}{\partial t} = -\frac{\mathrm{d}H}{\mathrm{d}t}}
\end{aligned}$$

5.2

For two functions $A(q,p)$ and $B(q,p)$, the Poisson bracket between them can be shown to be anti-symmetric as follows

$$\begin{aligned}
\{A,B\} &= \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \\
&= \frac{\partial B}{\partial p}\frac{\partial A}{\partial q} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} \\
&= -\frac{\partial B}{\partial q}\frac{\partial A}{\partial p} + \frac{\partial B}{\partial p}\frac{\partial A}{\partial q} \\
&\boxed{\{A,B\} = -\{B,A\}}
\end{aligned}$$

For the Jacobi identity, we can consider an infinitesimal canonical transformation generated by some function $C(q,p)$. The change in the Poisson bracket will be

$$\delta\{A,B\} = \epsilon\{\{A,B\},C\}$$

where ϵ is an infinitesimal. This can also be written in terms of the individual variations of A and B

$$\begin{aligned}
\delta\{A,B\} &= \{\delta A, B\} + \{A, \delta B\} \\
&= \epsilon\{\{A,C\},B\} + \epsilon\{A,\{B,C\}\} \\
\delta\{A,B\} &= -\epsilon\{\{C,A\},B\} - \epsilon\{\{B,C\},A\}
\end{aligned}$$

Putting these changes together, we obtain

$$\boxed{\{\{A,B\},C\} + \{\{C,A\},B\} + \{\{B,C\},A\} = 0}$$

As for quantum operators, we also have anti-symmetry

$$\begin{aligned}
[\hat{A},\hat{B}] &= \hat{A}\hat{B} - \hat{B}\hat{A} \\
&= -\left(\hat{B}\hat{A} - \hat{A}\hat{B}\right) \\
\boxed{[\hat{A},\hat{B}]} &\boxed{= -[\hat{B},\hat{A}]}
\end{aligned}$$

and the Jacobi identity

$$\begin{aligned}
[[\hat{A},\hat{B}],\hat{C}] + [[\hat{C},\hat{A}],\hat{B}] + [[\hat{B},\hat{C}],\hat{A}] &= [\hat{A}\hat{B} - \hat{B}\hat{A},\hat{C}] + [\hat{C}\hat{A} - \hat{A}\hat{C},\hat{B}] + [\hat{B}\hat{C} - \hat{C}\hat{B},\hat{A}] \\
&= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + \hat{C}(\hat{A}\hat{B} - \hat{B}\hat{A}) + (\hat{C}\hat{A} - \hat{A}\hat{C})\hat{B} \\
&\quad + \hat{B}(\hat{C}\hat{A} - \hat{A}\hat{C}) + (\hat{B}\hat{C} - \hat{C}\hat{B})\hat{A} - \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) \\
&= (\hat{A}\hat{B}\hat{C} - \hat{A}\hat{B}\hat{C}) + (\hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C}) + (\hat{A}\hat{C}\hat{B} - \hat{A}\hat{C}\hat{B}) \\
&\quad + (\hat{B}\hat{C}\hat{A} - \hat{B}\hat{C}\hat{A}) + (\hat{C}\hat{A}\hat{B} - \hat{C}\hat{A}\hat{B}) + (\hat{C}\hat{B}\hat{A} - \hat{C}\hat{B}\hat{A}) \\
\boxed{[[\hat{A},\hat{B}],\hat{C}] + [[\hat{C},\hat{A}],\hat{B}] + [[\hat{B},\hat{C}],\hat{A}]} &\boxed{= 0}
\end{aligned}$$

5.3

The commutator of two Hermitian operators $\hat{A} = \hat{A}^\dagger$ and $\hat{B} = \hat{B}^\dagger$ can be shown to be anti-symmetric as follows

$$\begin{aligned}
[\hat{A},\hat{B}]^\dagger &= (\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
&= \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger \\
&= \hat{B}\hat{A} - \hat{A}\hat{B} \\
&- (\hat{A}\hat{B} - \hat{B}\hat{A}) \\
\boxed{[\hat{A},\hat{B}]^\dagger} &\boxed{= -[\hat{A},\hat{B}]}
\end{aligned}$$

5.4

For the Lagrangian

$$L = \frac{-mc^2}{\gamma}$$

in the limit $v \ll c$, the Lagrangian can be written as

$$\begin{aligned}
L &= -mc^2\sqrt{1 - \frac{v^2}{c^2}} \\
&\approx -mc^2\left(1 - \frac{v^2}{2c^2}\right) \\
\boxed{L} &\boxed{= -mc^2 + \frac{1}{2}mv^2}
\end{aligned}$$

the momentum as

$$\begin{aligned}
p &= \frac{\partial L}{\partial v} \\
&= mc^2\frac{v/c^2}{\sqrt{1 - (v/c)^2}} \\
&\approx \frac{m}{v}\left(\frac{v}{c}\right)^2\left(1 + \frac{v^2}{2c^2}\right) \\
\boxed{p} &\boxed{= mv}
\end{aligned}$$

and the Hamiltonian as

$$\begin{aligned}
H &= pv - L \\
&= mc^2\left(\frac{v}{c}\right)^2 \frac{1}{\sqrt{1-(v/c)^2}} + mc^2\sqrt{1-\left(\frac{v}{c}\right)^2} \\
&\approx mc^2\left[\left(\frac{v}{c}\right)^2\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2\right)+\left(1-\frac{1}{2}\left(\frac{v}{c}\right)^2\right)\right] \\
&\boxed{H = mc^2 + \frac{1}{2}mv^2}
\end{aligned}$$

5.5

The integral shown below can be extremized by first writing it as a functional as follows

$$\begin{aligned}
S &= \int_a^b \mathrm{d}s \\
&= \int_a^b \sqrt{c^2\,\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2} \\
&= c\int_{t_a}^{t_b} \sqrt{1-\frac{1}{c^2}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
S[\dot{\mathbf{x}}] &= c\int_{t_a}^{t_b} \sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}\,\mathrm{d}t
\end{aligned}$$

The condition for extremization will then be that the integrand satisfies the Euler-Lagrange equation

$$\begin{gathered}
\frac{\partial}{\partial \mathbf{x}}\left(c\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\mathbf{x}}}\left(c\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}\right) = 0 \\
\frac{\partial}{\partial \dot{\mathbf{x}}}\left(c\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}}\right) = \alpha \\
\frac{\dot{\mathbf{x}}}{c}\left(1-\frac{\dot{\mathbf{x}}^2}{c^2}\right)^{-1/2} = \alpha \\
\dot{\mathbf{x}}^2 = \alpha^2 c^2\left(1-\frac{\dot{\mathbf{x}}^2}{c^2}\right) \\
\dot{\mathbf{x}} = \frac{\alpha}{\sqrt{1+\alpha^2}}c = mc \\
\boxed{\mathbf{x}(t) = m(ct) + b}
\end{gathered}$$

5.6

For the Lagrangian

$$L = -mc^2\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}} + q\mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}} - qV(\mathbf{x})$$

the equations of motion given by are

$$
\begin{aligned}
\frac{\partial L}{\partial \mathbf{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{x}}} &= 0 \\
q\left(\boldsymbol{\nabla}(\mathbf{A}\cdot\dot{\mathbf{x}}) - \boldsymbol{\nabla}V\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\dot{\mathbf{x}} + q\mathbf{A}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\dot{\mathbf{x}}\right) = q\left[(\mathbf{A}\cdot\boldsymbol{\nabla})\dot{\mathbf{x}} + (\dot{\mathbf{x}}\cdot\boldsymbol{\nabla})\mathbf{A} + \dot{\mathbf{x}}\times\boldsymbol{\nabla}\times\mathbf{A} + \mathbf{A}\times\boldsymbol{\nabla}\times\dot{\mathbf{x}} - \boldsymbol{\nabla}V - \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right]& \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\dot{\mathbf{x}}\right) = q\left[(\dot{\mathbf{x}}\cdot\boldsymbol{\nabla})\mathbf{A} + \dot{\mathbf{x}}\times\boldsymbol{\nabla}\times\mathbf{A} - \boldsymbol{\nabla}V - \frac{\partial\mathbf{A}}{\partial t} - (\dot{\mathbf{x}}\cdot\boldsymbol{\nabla})\mathbf{A}\right]& \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\dot{\mathbf{x}}\right) = q\left[-\boldsymbol{\nabla}V - \frac{\partial\mathbf{A}}{\partial t} + \dot{\mathbf{x}}\times\boldsymbol{\nabla}\times\mathbf{A}\right]& \\
\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\dot{\mathbf{x}}\right) = q\left(\mathbf{E} + \dot{\mathbf{x}}\times\mathbf{B}\right)}&
\end{aligned}
$$

5.7

For the Lagrangian

$$L = -mc^2\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}} + q\mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}} - qV(\mathbf{x})$$

in the limit $v \ll c$, the conjugate momentum is

$$
\begin{aligned}
\mathbf{p} &= \frac{\partial L}{\partial \dot{\mathbf{x}}} \\
&\approx \frac{\partial}{\partial \dot{\mathbf{x}}}\left(-mc^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 + q\mathbf{A}\cdot\dot{\mathbf{x}} - qV\right) \\
&\boxed{\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A}}
\end{aligned}
$$

and the energy is

$$
\begin{aligned}
H &= \mathbf{p}\cdot\dot{\mathbf{x}} - L \\
&= m\dot{\mathbf{x}}^2 + q\mathbf{A}\cdot\dot{\mathbf{x}} - L \\
&= mc^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 + qV \\
&\boxed{H = mc^2 + \frac{(\mathbf{p}-q\mathbf{A})^2}{2m} + qV}
\end{aligned}
$$

5.8

The invariant $\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$ can be expressed in terms of the electric and magnetic fields as follows

$$
\begin{aligned}
\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} &= 2(F_{01}-F_{10})(F_{23}-F_{32}) - 2(F_{02}-F_{20})(F_{13}-F_{31}) + 2(F_{03}-F_{30})(F_{12}-F_{21}) \\
&= 2(2E_1)(-2B_1) - 2(2E_2)(2B_2) + 2(2E_3)(-2B_3) \\
&\boxed{\varepsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = -8\,\mathbf{E}\cdot\mathbf{B}}
\end{aligned}
$$

which shows that $\mathbf{E}\cdot\mathbf{B}$ is also invariant.

5.9

The expression $\partial_\mu F^{\mu\nu} = J^\nu$ yields the following Maxwell equations

$$\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = J^0$$
$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{E} = \rho}$$

$$\begin{aligned} -\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2 &= J^1 \\ -\partial_0 E_2 - \partial_1 B_3 + \partial_3 B_1 &= J^2 \\ -\partial_0 E_3 + \partial_1 B_2 - \partial_2 B_1 &= J^3 \end{aligned}$$
$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}}$$

The expression $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ yields the remaining two Maxwell equations

$$\begin{aligned} \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0 \\ -\partial_1 B_1 - \partial_2 B_2 - \partial_3 B_3 &= 0 \end{aligned}$$
$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0}$$

$$\begin{aligned} \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} &= 0 \\ -\partial_0 B_1 - \partial_1 E_2 + \partial_2 E_1 &= 0 \end{aligned}$$

$$\begin{aligned} \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} &= 0 \\ \partial_0 B_2 - \partial_1 E_3 + \partial_3 E_1 &= 0 \end{aligned}$$

$$\begin{aligned} \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} &= 0 \\ -\partial_0 B_3 - \partial_2 E_3 + \partial_3 E_2 &= 0 \end{aligned}$$
$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

5.10

Since the electromagnetic field strength tensor is anti-symmetric $F_{\mu\nu} = -F_{\nu\mu}$, we can write

$$\begin{aligned} \partial_\beta \partial_\alpha F^{\alpha\beta} &= \frac{1}{2}\partial_\beta \partial_\alpha \left(F^{\alpha\beta} - F^{\beta\alpha}\right) \\ &= \frac{1}{2}\left(\partial_\beta \partial_\alpha F^{\alpha\beta} - \partial_\beta \partial_\alpha F^{\beta\alpha}\right) \\ &= \frac{1}{2}\left(\partial_\beta \partial_\alpha F^{\alpha\beta} - \partial_\alpha \partial_\beta F^{\beta\alpha}\right) \\ &= \frac{1}{2}\left(\partial_\beta \partial_\alpha F^{\alpha\beta} - \partial_\beta \partial_\alpha F^{\alpha\beta}\right) \\ \Aboxed{\partial_\beta \partial_\alpha F^{\alpha\beta} &= 0} \end{aligned}$$

Since $\partial_\mu F^{\mu\nu} = J^\nu$, this implies that

$$\partial_\mu J^\mu = 0$$
$$\partial_0 J^0 + \partial_i J^i = 0$$
$$\boxed{\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{J}}$$

which is the local charge continuity equation.