

Chapter 3

3.1

Using the definition of the dirac delta

$$\delta^{(3)}(\mathbf{x} - \mathbf{y}) = \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}$$

we have for bosons

$$\begin{aligned} \frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] &= \frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} \delta_{\mathbf{p}, \mathbf{q}} \\ &= \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \end{aligned}$$

$\frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger] = \delta^{(3)}(\mathbf{x} - \mathbf{y})$

and for fermions

$$\begin{aligned} \frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} \{ \hat{c}_{\mathbf{p}}, \hat{c}_{\mathbf{q}}^\dagger \} &= \frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} \delta_{\mathbf{p}, \mathbf{q}} \\ &= \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \end{aligned}$$

$\frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} \{ \hat{c}_{\mathbf{p}}, \hat{c}_{\mathbf{q}}^\dagger \} = \delta^{(3)}(\mathbf{x} - \mathbf{y})$

3.2

For the simple harmonic oscillator, we have

$$\begin{aligned} [\hat{a}, (\hat{a}^\dagger)^n] |m\rangle &= \hat{a}(\hat{a}^\dagger)^n |m\rangle - (\hat{a}^\dagger)^n \hat{a} |m\rangle \\ &= \hat{a} \sqrt{(m+1)(m+2) \dots (m+n)} |m+n\rangle - (\hat{a}^\dagger)^n \sqrt{m} |m-1\rangle \\ &= [(m+n)-m] \sqrt{(m+1) \dots (m+n-1)} |m+n-1\rangle \\ &= n \sqrt{(m+1) \dots (m+n-1)} |m+n-1\rangle \\ [\hat{a}, (\hat{a}^\dagger)^n] |m\rangle &= n(\hat{a}^\dagger)^{n-1} |m\rangle \quad \forall |m\rangle \end{aligned}$$

$[\hat{a}, (\hat{a}^\dagger)^n] = n(\hat{a}^\dagger)^{n-1}$

as well as

$$\begin{aligned}\langle 0 | (\hat{a})^n (\hat{a}^\dagger)^m | 0 \rangle &= \langle n | \sqrt{n!} \sqrt{m!} | m \rangle \\ &= \sqrt{n! m!} \delta_{n,m} \\ \boxed{\langle 0 | (\hat{a})^n (\hat{a}^\dagger)^m | 0 \rangle = n! \delta_{n,m}}\end{aligned}$$

and

$$\begin{aligned}\langle m | \hat{a}^\dagger | n \rangle &= \sqrt{n+1} \langle m | n+1 \rangle \\ \boxed{\langle m | \hat{a}^\dagger | n \rangle = \sqrt{n+1} \delta_{m,n+1}}\end{aligned}$$

$$\begin{aligned}\langle m | \hat{a} | n \rangle &= \sqrt{n} \langle m | n-1 \rangle \\ \boxed{\langle m | \hat{a} | n \rangle = \sqrt{n} \delta_{m,n-1}}\end{aligned}$$

3.3

For the three-dimensional harmonic oscillator

$$\hat{H} = \frac{1}{2m} (\hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2) + \frac{1}{2} m \omega^2 (\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2)$$

we can define creation and annihilation operators

$$\hat{a}_i = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x}_i + \frac{i}{m\omega} \hat{p}_i \right), \quad \hat{a}_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x}_i - \frac{i}{m\omega} \hat{p}_i \right)$$

and write the Hamiltonian as

$$\begin{aligned}\hat{H} &= \hat{H}_1 + \hat{H}_2 + \hat{H}_3 \\ &= \hbar\omega \left(\hat{a}_1^\dagger \hat{a}_1 + \frac{1}{2} \right) + \hbar\omega \left(\hat{a}_2^\dagger \hat{a}_2 + \frac{1}{2} \right) + \hbar\omega \left(\hat{a}_3^\dagger \hat{a}_3 + \frac{1}{2} \right) \\ \boxed{\hat{H} = \hbar\omega \sum_{i=1}^3 \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right)}\end{aligned}$$

Defining new creation and annihilation operators

$$\hat{b}_1^\dagger = -\frac{1}{\sqrt{2}} \left(\hat{a}_1^\dagger + i\hat{a}_2^\dagger \right), \quad b_0^\dagger = \hat{a}_3^\dagger, \quad \hat{b}_{-1}^\dagger = \frac{1}{\sqrt{2}} \left(\hat{a}_1^\dagger - i\hat{a}_2^\dagger \right)$$

we see that

$$\begin{aligned}[\hat{b}_1, \hat{b}_1^\dagger] &= \frac{1}{2} \left((\hat{a}_1 - i\hat{a}_2) (\hat{a}_1^\dagger + i\hat{a}_2^\dagger) - (\hat{a}_1^\dagger + i\hat{a}_2^\dagger) (\hat{a}_1 - i\hat{a}_2) \right) = \frac{1}{2} ([\hat{a}_1, \hat{a}_1^\dagger] + [\hat{a}_2, \hat{a}_2^\dagger]) = 1 \\ [\hat{b}_0, \hat{b}_0^\dagger] &= [\hat{a}_3, \hat{a}_3^\dagger] = 1 \\ [\hat{b}_{-1}, \hat{b}_{-1}^\dagger] &= \frac{1}{2} \left((\hat{a}_1 + i\hat{a}_2) (\hat{a}_1^\dagger - i\hat{a}_2^\dagger) - (\hat{a}_1^\dagger - i\hat{a}_2^\dagger) (\hat{a}_1 + i\hat{a}_2) \right) = \frac{1}{2} ([\hat{a}_1, \hat{a}_1^\dagger] + [\hat{a}_2, \hat{a}_2^\dagger]) = 1 \\ [\hat{b}_1, \hat{b}_{-1}^\dagger] &= \frac{1}{2} \left((\hat{a}_1 - i\hat{a}_2) (\hat{a}_1^\dagger - i\hat{a}_2^\dagger) - (\hat{a}_1^\dagger - i\hat{a}_2^\dagger) (\hat{a}_1 - i\hat{a}_2) \right) = -\frac{1}{2} ([\hat{a}_1, \hat{a}_1^\dagger] - [\hat{a}_2, \hat{a}_2^\dagger]) = 0 \\ \boxed{[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}}\end{aligned}$$

Inverting the above definitions

$$\hat{a}_1^\dagger = -\frac{1}{\sqrt{2}} \left(\hat{b}_1^\dagger - \hat{b}_{-1}^\dagger \right), \quad \hat{a}_2^\dagger = \frac{i}{\sqrt{2}} \left(\hat{b}_1^\dagger + \hat{b}_{-1}^\dagger \right), \quad \hat{a}_3^\dagger = \hat{b}_0^\dagger$$

the Hamiltonian can be expressed as

$$\begin{aligned}
 \hat{H} &= \hbar\omega \sum_{i=1}^3 \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right) \\
 &= \frac{1}{2} \hbar\omega \left[(\hat{b}_1^\dagger - \hat{b}_{-1}^\dagger) (\hat{b}_1 - \hat{b}_{-1}) + 1 + (\hat{b}_1^\dagger + \hat{b}_{-1}^\dagger) (\hat{b}_1 + \hat{b}_{-1}) + 1 + \hat{b}_0^\dagger \hat{b}_0 + 1 \right] \\
 &= \hbar\omega \left[\left(\hat{b}_1^\dagger \hat{b}_1 + \frac{1}{2} \right) + \left(\hat{b}_0^\dagger \hat{b}_0 + \frac{1}{2} \right) + \left(\hat{b}_{-1}^\dagger \hat{b}_{-1} + \frac{1}{2} \right) \right] \\
 \boxed{\hat{H} = \hbar\omega \sum_{m=-1}^1 \left(\hat{b}_m^\dagger \hat{b}_m + \frac{1}{2} \right)}
 \end{aligned}$$

Defining angular momentum as

$$\hat{L}^i = -i\hbar\epsilon^{ijk}\hat{a}_j^\dagger \hat{a}_k$$

we can express L^3 it in terms of the new operators as

$$\begin{aligned}
 \hat{L}^3 &= -i\hbar(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \\
 &= \frac{\hbar}{2} \left[(\hat{b}_1^\dagger - \hat{b}_{-1}^\dagger) (\hat{b}_1 + \hat{b}_{-1}) + (\hat{b}_1^\dagger + \hat{b}_{-1}^\dagger) (\hat{b}_1 - \hat{b}_{-1}) \right] \\
 &= \hbar(\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_{-1}^\dagger \hat{b}_{-1}) \\
 \boxed{\hat{L}^3 = \hbar \sum_{m=-1}^1 m \hat{b}_m^\dagger \hat{b}_m}
 \end{aligned}$$

3.4

For two particles, we have

$$\begin{aligned}
 \Psi(r_1, r_2) &= \frac{1}{\sqrt{2!}} (\psi_1(r_1)\psi_2(r_2) - \psi_1(r_2)\psi_2(r_1)) \\
 &= \frac{1}{\sqrt{2!}} \begin{vmatrix} \psi_1(r_1) & \psi_1(r_2) \\ \psi_2(r_1) & \psi_2(r_2) \end{vmatrix} \\
 &= \frac{1}{\sqrt{2!}} \epsilon^{1,i_1,i_2} \psi_{i_1}(r_1) \psi_{i_2}(r_2)
 \end{aligned}$$

For three particles, we have

$$\begin{aligned}
 \Psi(r_1, r_2, r_3) &= \frac{1}{\sqrt{3!}} \epsilon^{1,i_1,i_2,i_3} \psi_{i_1}(r_1) \psi_{i_2}(r_2) \psi_{i_3}(r_3) \\
 &= \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_1(r_1) & \psi_1(r_2) & \psi_1(r_3) \\ \psi_2(r_1) & \psi_2(r_2) & \psi_2(r_3) \\ \psi_3(r_1) & \psi_3(r_2) & \psi_3(r_3) \end{vmatrix}
 \end{aligned}$$

For n -particles, the generalization is the Slater determinant

$$\begin{aligned}
 \Psi(r_1, \dots, r_n) &= \frac{1}{\sqrt{n!}} \epsilon^{1,i_1, \dots, i_n} \psi_{i_1}(r_1) \dots \psi_{i_n}(r_n) \\
 \Psi(r_1, \dots, r_n) &= \begin{vmatrix} \psi_1(r_1) & \psi_1(r_2) & \dots & \psi_1(r_n) \\ \psi_2(r_1) & \psi_2(r_2) & \dots & \psi_2(r_n) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n(r_1) & \psi_n(r_2) & \dots & \psi_n(r_n) \end{vmatrix}
 \end{aligned}$$