

# Chapter 2

## 2.1

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For creation and annihilation operators defined by

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right)$$

we have

$$[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$$

as well as

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \frac{m\omega}{2\hbar} \left[ \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right) - \left( \hat{x} - \frac{i}{m\omega} \hat{p} \right) \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) \right] \\ &= \frac{m\omega}{2\hbar} \left( \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega} (\hat{x}\hat{p} - \hat{p}\hat{x}) - \hat{x}^2 - \frac{\hat{p}^2}{m^2\omega^2} - \frac{i}{m\omega} (\hat{x}\hat{p} - \hat{p}\hat{x}) \right) \\ &= \frac{1}{i\hbar} [\hat{x}, \hat{p}] \\ [\hat{a}, \hat{a}^\dagger] &= 1 \end{aligned}$$

We can invert the above definitions to obtain

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}} (\hat{a} - \hat{a}^\dagger)$$

Plugging these into the harmonic oscillator Hamiltonian, we obtain

$$\begin{aligned} \hat{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \\ &= -\frac{1}{2m} \left( \frac{m\hbar\omega}{2} \right) (\hat{a} - \hat{a}^\dagger)^2 + \frac{1}{2} m\omega^2 \left( \frac{\hbar}{2m\omega} \right) (\hat{a} + \hat{a}^\dagger)^2 \\ &= \frac{\hbar\omega}{4} \left[ (\hat{a} + \hat{a}^\dagger)^2 - (\hat{a} - \hat{a}^\dagger)^2 \right] \\ &= \frac{\hbar\omega}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) \\ \hat{H} &= \hbar\omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) = \hbar\omega \left( \hat{n} + \frac{1}{2} \right) \end{aligned}$$

## 2.2

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For the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}^4 = \hat{H}_0 + \lambda\hat{x}^4$$

where  $\lambda \ll 1$ , the energy levels to first order in  $\lambda$  are given by

$$\begin{aligned} E_n &= \langle n | \hat{H}_0 | n \rangle + \langle n | \lambda \hat{x}^4 | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \langle n | \hat{x}^4 | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (\hat{a} + \hat{a}^\dagger)^4 | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | ((\hat{a})^2 + (\hat{a}^\dagger)^2 + 2\hat{n} + 1)^2 | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (4\hat{n}^2 + 1 + (\hat{a})^2(\hat{a}^\dagger)^2 + (\hat{a}^\dagger)^2(\hat{a})^2 + 4\hat{n}) | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (4\hat{n}^2 + 4\hat{n} + 1 + \hat{a}(1 + \hat{n})\hat{a}^\dagger + \hat{a}^\dagger\hat{n}\hat{a}) | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (4\hat{n}^2 + 4\hat{n} + 1 + \hat{n} + 1 + (\hat{n}\hat{a} + \hat{a})\hat{a}^\dagger + \hat{a}^\dagger(\hat{a}\hat{n} - \hat{a})) | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (4\hat{n}^2 + 5\hat{n} + 2 + \hat{n}(\hat{n} + 1) + \hat{n} + 1 + \hat{n}^2 - \hat{n}) | n \rangle \\ &= \hbar\omega \left( n + \frac{1}{2} \right) + \lambda \left( \frac{\hbar}{2m\omega} \right)^2 \langle n | (6\hat{n}^2 + 6\hat{n} + 3) | n \rangle \end{aligned}$$

$$E_n = \hbar\omega \left( n + \frac{1}{2} \right) + \frac{3\lambda}{4} \left( \frac{\hbar}{m\omega} \right)^2 (2n^2 + 2n + 1)$$

## 2.3

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Using the equations

$$x_j = \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k e^{ikja}, \quad \tilde{x}_k = \sqrt{\frac{\hbar}{2m\omega_k}} (\hat{a}_k + \hat{a}_{-k}^\dagger)$$

we have

$$x_j = \frac{1}{\sqrt{N}} \left( \frac{\hbar}{m} \right)^{1/2} \sum_k \frac{1}{(2\omega_k)^{1/2}} (\hat{a}_k e^{ikja} + \hat{a}_{-k}^\dagger e^{ikja})$$

Since the sum over  $k$  is symmetric about  $k = 0$

$$k = \frac{2\pi m}{Na}, \quad m \in \left[ -\frac{N}{2}, \frac{N}{2} \right]$$

and  $\omega_{-k} = \omega_k$  for this system, we can re-index the second term in the above expression to read

$$x_j = \frac{1}{\sqrt{N}} \left( \frac{\hbar}{m} \right)^{1/2} \sum_k \frac{1}{(2\omega_k)^{1/2}} (\hat{a}_k e^{ikja} + \hat{a}_k^\dagger e^{-ikja})$$

## 2.4

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Using the three equations

$$\hat{a}|0\rangle = 0, \quad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right), \quad \langle x|\hat{p}|\psi\rangle = -i\hbar \frac{d}{dx} \langle x|\psi\rangle$$

we can combine them to write

$$\begin{aligned} \langle x|\hat{a}|0\rangle &= \sqrt{\frac{m\omega}{2\hbar}} \langle x| \left( \hat{x} + \frac{i}{m\omega} \hat{p} \right) |0\rangle = 0 \\ &\boxed{\left( x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \langle x|0\rangle = 0} \end{aligned}$$

This differential equation has the normalizable solution

$$\langle x|0\rangle = A e^{-m\omega x^2/2\hbar}$$

Normalized to unity, we obtain

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\langle x|0\rangle|^2 dx \\ &= |A|^2 \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} dx \\ 1 &= |A|^2 \sqrt{\frac{\hbar\pi}{m\omega}} \\ A &= \left( \frac{m\omega}{\hbar\pi} \right)^{1/4} \end{aligned}$$

which gives the normalized solution

$$\boxed{\langle x|0\rangle = \left( \frac{m\omega}{\hbar\pi} \right)^{1/4} e^{-m\omega x^2/2\hbar}}$$