Chapter 12

12.1

The Hamiltonian for a complex scalar field has the form

$$H = \int \mathrm{d}^3x \left(\dot{\psi}^\dagger\dot{\psi} + \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi + m^2\psi^\dagger\psi\right)$$

In terms of creation and annihilation operators

$$\hat{\psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \frac{1}{(2E_{\mathbf{p}})^{1/2}} \left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)$$

the Hamiltonian can be express as

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\int \frac{\mathrm{d}^3x\,\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \Big[E_{\mathbf{p}}E_{\mathbf{q}}\left(\hat{a}^\dagger_{\mathbf{p}}e^{ip\cdot x} - \hat{b}_{\mathbf{p}}e^{-ip\cdot x}\right)\left(\hat{a}_{\mathbf{q}}e^{-iq\cdot x} - \hat{b}^\dagger_{\mathbf{q}}e^{iq\cdot x}\right) + \mathbf{p}\cdot\mathbf{q} \\
&\quad \times\left(\hat{a}^\dagger_{\mathbf{p}}e^{ip\cdot x} - \hat{b}_{\mathbf{p}}e^{-ip\cdot x}\right)\left(\hat{a}_{\mathbf{q}}e^{-iq\cdot x} - \hat{b}^\dagger_{\mathbf{q}}e^{iq\cdot x}\right) + m^2\left(\hat{a}^\dagger_{\mathbf{p}}e^{ip\cdot x} + \hat{b}_{\mathbf{p}}e^{-ip\cdot x}\right)\left(\hat{a}_{\mathbf{q}}e^{-iq\cdot x} + \hat{b}^\dagger_{\mathbf{q}}e^{iq\cdot x}\right)\Big] \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3x\,\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \Big[\left(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} + m^2\right)\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}e^{i(p-q)\cdot x} + \hat{b}_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{q}}e^{-i(p-q)\cdot x}\right) \\
&\quad + \left(-E_{\mathbf{p}}E_{\mathbf{q}} - \mathbf{p}\cdot\mathbf{q} + m^2\right)\left(\hat{a}^\dagger_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{q}}e^{i(p+q)\cdot x} + \hat{b}_{\mathbf{p}}\hat{a}_{\mathbf{q}}e^{-i(p+q)\cdot x}\right)\Big] \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \Big[\left(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} + m^2\right)\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}e^{i(E_{\mathbf{p}}-E_{\mathbf{q}})t} + \hat{b}_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{q}}e^{-i(E_{\mathbf{p}}-E_{\mathbf{q}})t}\right)\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&\quad + \left(-E_{\mathbf{p}}E_{\mathbf{q}} - \mathbf{p}\cdot\mathbf{q} + m^2\right)\left(\hat{a}^\dagger_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{q}}e^{i(E_{\mathbf{p}}+E_{\mathbf{q}})t} + \hat{b}_{\mathbf{p}}\hat{a}_{\mathbf{q}}e^{-i(E_{\mathbf{p}}+E_{\mathbf{q}})t}\right)\delta^{(3)}(\mathbf{p}+\mathbf{q})\Big] \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{E_{\mathbf{p}}} \Big[\left(E^2_{\mathbf{p}} + \mathbf{p}^2 + m^2\right)\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{b}_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{p}}\right) + \left(-E^2_{\mathbf{p}} + \mathbf{p}^2 + m^2\right)\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}e^{2iE_{\mathbf{p}}t} + \hat{b}_{\mathbf{p}}\hat{a}_{-\mathbf{p}}e^{-2iE_{\mathbf{p}}t}\right)\Big] \\
H &= \int \mathrm{d}^3p\, E_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{b}_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{p}}\right)
\end{aligned}$$

Normal-ordered, we obtain

$$\boxed{N[\hat{H}] = \int \mathrm{d}^3p\, E_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{b}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right) = \int \mathrm{d}^3p\, E_{\mathbf{p}}\left(\hat{n}^{(a)}_{\mathbf{p}} + \hat{n}^{(b)}_{\mathbf{p}}\right)}$$

12.2

For the complex scalar field, the commutator $[\hat{\psi}(x), \hat{\psi}^\dagger(y)]$ is given by

$$\begin{aligned}
[\hat{\psi}(x), \hat{\psi}^\dagger(y)] &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \left[\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{b}^\dagger_{\mathbf{p}}e^{ip\cdot x}\right), \left(\hat{a}^\dagger_{\mathbf{q}}e^{iq\cdot y} + \hat{b}_{\mathbf{q}}e^{-iq\cdot y}\right)\right] \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right]e^{-i(p\cdot x - q\cdot y)} + \left[\hat{b}^\dagger_{\mathbf{p}}, \hat{b}_{\mathbf{q}}\right]e^{i(p\cdot x - q\cdot y)}\right) \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \left(e^{-i(p\cdot x - q\cdot y)} - e^{i(p\cdot x - q\cdot y)}\right)\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
[\hat{\psi}(x), \hat{\psi}^\dagger(y)] &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)
\end{aligned}$$

For equal times, this commutator vanishes due to the properties of space-like intervals noted in Exercise 11.1

$$\boxed{[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = 0, \quad (x-y) \text{ space-like}}$$

For the non-relativistic limit, the field is given by

$$\hat{\Psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \hat{a}_{\mathbf{p}} e^{-ip\cdot x}$$

In this case, the commutator $[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)]$ is given by

$$\begin{aligned}
[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)] &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] e^{-i(p\cdot x - q\cdot y)} \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} e^{-i(p\cdot x - q\cdot y)}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)] &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{-ip\cdot(x-y)}
\end{aligned}$$

For equal times, this takes the form

$$\boxed{[\hat{\Psi}(x), \hat{\Psi}^\dagger(y)] = \delta^{(3)}(\mathbf{x}-\mathbf{y})}$$

12.3

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_1)^2 - \frac{1}{2}m^2\varphi_1^2 + \frac{1}{2}(\partial_\mu\varphi_2)^2 - \frac{1}{2}m^2\varphi_2^2 - g(\varphi_1+\varphi_2)^2$$

which has the internal symmetry

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

the corresponding changes in the fields are given by

$$D\varphi_1 = \left.\frac{\partial\varphi_1}{\partial\vartheta}\right|_{\vartheta=0} = -\varphi_2, \quad D\varphi_2 = \left.\frac{\partial\varphi_2}{\partial\vartheta}\right|_{\vartheta=0} = \varphi_1$$

Therefore, we have

$$\boxed{\begin{aligned}[\hat{Q}_N, \hat{\varphi}_1] &= -iD\hat{\varphi}_1 = i\hat{\varphi}_2\\ [\hat{Q}_N, \hat{\varphi}_2] &= -iD\hat{\varphi}_2 = -i\hat{\varphi}_1\end{aligned}}$$

as well as

$$\begin{aligned}
[\hat{Q}_N, \hat{\psi}] &= \frac{1}{\sqrt{2}}\Big([\hat{Q}_N, \hat{\varphi}_1] + i[\hat{Q}_N, \hat{\varphi}_2]\Big) \\
&= \frac{1}{\sqrt{2}}(i\hat{\varphi}_2 + \hat{\varphi}_1) \\
\boxed{[\hat{Q}_N, \hat{\psi}] &= \hat{\psi}}
\end{aligned}$$

12.4

For the Lagrangian

$$\mathcal{L} = \frac{i}{2}\partial_0\rho - \rho\partial_0\vartheta - \frac{1}{2m}\left[\frac{1}{4\rho}(\boldsymbol{\nabla}\rho)^2 + \rho(\boldsymbol{\nabla}\vartheta)^2\right] - \frac{g}{2}\rho$$

with Noether current

$$J_N^\mu = \begin{pmatrix} -\rho(x) & -\frac{\rho(x)}{m}\boldsymbol{\nabla}\vartheta \end{pmatrix}$$

the Noether charge is given by

$$\begin{aligned}
\hat{Q} &= \int \mathrm{d}^3x\, \hat{J}_N^0 \\
&= -\int \mathrm{d}^3x\, \hat{\rho}(x) \\
\hat{Q} &= -\hat{N}
\end{aligned}$$

The change in the variable ϑ is simply $D\vartheta = 1$, which yields

$$\begin{aligned}
[\hat{Q}, \hat{\vartheta}] &= -iD\vartheta \\
\boxed{[\hat{N}, \hat{\vartheta}] &= i}
\end{aligned}$$

12.5

The non-relativistic limit of a complex scalar field with no external potential is given by

$$\mathcal{L} = i\Psi^\dagger(x)\partial_0\Psi(x) - \frac{1}{2m}\boldsymbol{\nabla}\Psi^\dagger(x)\cdot\boldsymbol{\nabla}\Psi(x)$$

The Euler-Lagrange equations of motion yield

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\Psi^\dagger} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^\dagger)}\right) &= 0 \\
i\partial_0\Psi + \frac{1}{2m}\nabla^2\Psi &= 0 \\
-\frac{1}{2m}\nabla^2\Psi &= i\partial_0\Psi
\end{aligned}$$

Identifying

$$E_{\mathbf{p}} \to i\partial_0, \quad \hat{\mathbf{p}} \to -i\boldsymbol{\nabla}$$

we have

$$\frac{\mathbf{p}^2}{2m}\Psi = E_{\mathbf{p}}\Psi$$

$$\boxed{E_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m}}$$

12.6

The complex scalar field

$$\mathcal{L} = i\Psi^\dagger(x)\partial_0\Psi(x) - \frac{1}{2m}\nabla\Psi^\dagger(x)\cdot\nabla\Psi(x)$$

has a $U(1)$ symmetry

$$\Psi \to e^{-i\alpha}\Psi = \Psi - i\alpha\Psi, \quad D\Psi = -i\Psi, \quad D\Psi^\dagger = i\Psi^\dagger$$

The associated Noether current is given by

$$\begin{aligned} J_N^\mu &= \Pi_a^\mu D\Psi^a \\ &= \Pi_\Psi^\mu D\Psi + \Pi_{\Psi^\dagger}^\mu D\Psi^\dagger \\ J_N^\mu &= -i\Pi_\Psi^\mu\Psi + i\Pi_{\Psi^\dagger}^\mu\Psi^\dagger \end{aligned}$$

with components

$$\boxed{\begin{aligned} J_N^0 &= \Psi^\dagger\Psi \\ J_N^i &= \frac{i}{2m}\left(\Psi\partial^i\Psi^\dagger - \Psi^\dagger\partial^i\Psi\right) \end{aligned}}$$

12.7

For an internal transformation operator

$$\hat{U}(\alpha) = e^{i\hat{Q}_{\mathrm{Nc}}\alpha}$$

we can write

$$\begin{aligned} \hat{U}^\dagger(\alpha)\hat{\psi}(x)\hat{U}(\alpha) &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\hat{\psi}(x)\left(1 + i\hat{Q}_{\mathrm{Nc}}\alpha + \frac{1}{2!}\left(i\hat{Q}_{\mathrm{Nc}}\alpha\right)^2 + \dots\right) \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\left(\hat{\psi} + (i\alpha)\hat{\psi}\hat{Q}_{\mathrm{Nc}} + \frac{1}{2!}(i\alpha)^2\hat{\psi}\hat{Q}_{\mathrm{Nc}}\hat{Q}_{\mathrm{Nc}} + \dots\right) \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\left(\hat{\psi} + (i\alpha)\left(\hat{Q}_{\mathrm{Nc}}\hat{\psi} + [\hat{\psi}, \hat{Q}_{\mathrm{Nc}}]\right) + \frac{(i\alpha)^2}{2}\left(\hat{Q}_{\mathrm{Nc}}\hat{\psi} + [\hat{\psi}, \hat{Q}_{\mathrm{Nc}}]\right)\hat{Q}_{\mathrm{Nc}} + \dots\right) \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\left(\hat{\psi} + (i\alpha)\left(\hat{Q}_{\mathrm{Nc}}\hat{\psi} + \hat{\psi}\right) + \frac{(i\alpha)^2}{2}\left(\hat{Q}_{\mathrm{Nc}}\hat{\psi} + \hat{\psi}\right)\hat{Q}_{\mathrm{Nc}} + \dots\right) \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\left(\hat{\psi} + (i\alpha)\left(\hat{Q}_{\mathrm{Nc}}\hat{\psi} + \hat{\psi}\right) + \frac{(i\alpha)^2}{2}\left(\hat{Q}_{\mathrm{Nc}}\hat{Q}_{\mathrm{Nc}}\hat{\psi} + \hat{Q}_{\mathrm{Nc}}[\hat{\psi}, \hat{Q}_{\mathrm{Nc}}]\right.\right. \\ &\quad \left.\left. + \hat{Q}_{\mathrm{Nc}}\hat{\psi} + [\hat{\psi}, \hat{Q}_{\mathrm{Nc}}]\right) + \dots\right) \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\left(\hat{\psi} + (i\alpha)\left(\hat{Q}_{\mathrm{Nc}}\hat{\psi} + \hat{\psi}\right) + \frac{(i\alpha)^2}{2}\left(\hat{Q}_{\mathrm{Nc}}\hat{Q}_{\mathrm{Nc}}\hat{\psi} + 2\hat{Q}_{\mathrm{Nc}}\hat{\psi} + \hat{\psi}\right) + \dots\right) \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}\left(1 + (i\alpha)\left(\hat{Q}_{\mathrm{Nc}} + 1\right) + \frac{(i\alpha)^2}{2}\left(\hat{Q}_{\mathrm{Nc}} + 1\right)^2 + \dots\right)\hat{\psi} \\ &= e^{-i\hat{Q}_{\mathrm{Nc}}\alpha}e^{i(\hat{Q}_{\mathrm{Nc}}+1)\alpha}\hat{\psi} \end{aligned}$$

$$\boxed{\hat{U}^\dagger(\alpha)\hat{\psi}(x)\hat{U}(\alpha) = e^{i\alpha}\hat{\psi}}$$