Chapter 11

11.1

The commutator \([\hat{\varphi}(x), \hat{\varphi}(y)]\) can be expressed in terms of creation and annihilation operators as follows

\[
[\hat{\varphi}(x), \hat{\varphi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} \left( [\hat{a}_p e^{-ip\cdot x} + \hat{a}_p^\dagger e^{ip\cdot x}], [\hat{a}_q e^{-i(p-x)\cdot y} + \hat{a}_q^\dagger e^{i(p-x)\cdot y}] \right)
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} \left( [\hat{a}_p, \hat{a}_q^\dagger] e^{-i(p-x)\cdot y} + [\hat{a}_q^\dagger, \hat{a}_q] e^{i(p-x)\cdot y} \right)
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} \left( e^{-i(p-x)\cdot y} - e^{i(p-x)\cdot y} \right) \delta^{(3)}(p - q)
\]

\[
[\hat{\varphi}(x), \hat{\varphi}(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left( e^{-ip\cdot (x-y)} - e^{ip\cdot (x-y)} \right)
\]

For space-like separations, there exists a Lorentz transformation that moves \(x - y\) into \(y - x\), and therefore the integrand vanishes

\[
[\hat{\varphi}(x), \hat{\varphi}(y)] = 0, \quad (x - y) \text{ space-like}
\]

11.2

At equal times \(x^0 = y^0\), the commutator \([\hat{\varphi}(x), \hat{\Pi}^0(y)]\) can be expressed in terms of creation and annihilation operators as follows

\[
[\hat{\varphi}(x), \hat{\Pi}^0(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} \left( [\hat{\varphi}(x), \hat{\Pi}^0(y)] \right)
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} \left( [\hat{a}_p, \hat{a}_q^\dagger] e^{-i(p-x)\cdot y} + [\hat{a}_q^\dagger, \hat{a}_q] e^{i(p-x)\cdot y} \right)
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2E_p E_q} \left( e^{-i(p-x)\cdot y} + e^{i(p-x)\cdot y} \right) \delta^{(3)}(p - q)
\]

\[
[\hat{\varphi}(x), \hat{\Pi}^0(y)] = i \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot (x-y)}
\]

At equal times, the separation \((x - y)\) is either light-like or space-like. Either way, we can freely move from \(x - y\) to \(y - x\) and write

\[
[\hat{\varphi}(x), \hat{\Pi}^0(y)] = i \delta^{(3)}(x - y), \quad x^0 = y^0
\]