Chapter 11

11.1

The commutator $[\hat{\varphi}(x), \hat{\varphi}(y)]$ can be expressed in terms of creation and annihilation operators as follows

$$\begin{aligned}
[\hat{\varphi}(x), \hat{\varphi}(y)] &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \left[\left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right), \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} + \hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot y}\right)\right] \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] e^{-i(p\cdot x - q\cdot y)} + \left[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\right] e^{i(p\cdot x - q\cdot y)}\right) \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{1}{2(E_{\mathbf{p}}E_{\mathbf{q}})^{1/2}} \left(e^{-i(p\cdot x - q\cdot y)} - e^{i(p\cdot x - q\cdot y)}\right) \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
[\hat{\varphi}(x), \hat{\varphi}(y)] &= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)
\end{aligned}$$

For space-like separations, there exists a Lorentz transformation that moves $x - y$ into $y - x$, and therefore the integrand vanishes

$$\boxed{[\hat{\varphi}(x), \hat{\varphi}(y)] = 0, \quad (x - y) \text{ space-like}}$$

11.2

At equal times $x^0 = y^0$, the commutator $[\hat{\varphi}(x), \hat{\Pi}^0(y)]$ can be expressed in terms of creation and annihilation operators as follows

$$\begin{aligned}
[\hat{\varphi}(x), \hat{\Pi}^0(y)] &= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{-i}{2} \left(\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}\right)^{1/2} \left[\left(\hat{a}_{\mathbf{p}} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}} e^{ip\cdot x}\right), \left(\hat{a}_{\mathbf{q}} e^{-iq\cdot y} - \hat{a}^\dagger_{\mathbf{q}} e^{iq\cdot y}\right)\right] \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{i}{2} \left(\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}\right)^{1/2} \left(\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] e^{-i(p\cdot x - q\cdot y)} - \left[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\right] e^{i(p\cdot x - q\cdot y)}\right) \\
&= \int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^3} \frac{i}{2} \left(\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}\right)^{1/2} \left(e^{-i(p\cdot x - q\cdot y)} + e^{i(p\cdot x - q\cdot y)}\right) \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\
[\hat{\varphi}(x), \hat{\Pi}^0(y)] &= \frac{i}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right)
\end{aligned}$$

At equal times, the separation $(x - y)$ is either light-like or space-like. Either way, we can freely move from $x - y$ to $y - x$ and write

$$[\hat{\varphi}(x), \hat{\Pi}^0(y)] = i \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

$$\boxed{[\hat{\varphi}(x), \hat{\Pi}^0(y)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}), \quad x^0 = y^0}$$