Chapter 10

10.1

The commutator in question can be calculated as follows

$$\begin{aligned}
[\varphi(x), P^\alpha] &= \int \mathrm{d}^3y \, [\varphi(x), T^{0\alpha}] \\
&= \int \mathrm{d}^3y \, [\varphi(x), \delta_0^\alpha \mathcal{H} + \delta_k^\alpha P^k] \\
[\varphi(x), P^\alpha] &= \delta_0^\alpha [\varphi(x), H] + \delta_k^\alpha \int \mathrm{d}^3y \left[\varphi(x), \pi(y)\partial^k_{(y)}\varphi(y)\right]
\end{aligned}$$

Using the Heisenberg equations of motion, the first term becomes

$$\begin{aligned}
[\varphi(x), P^\alpha] &= \delta_0^\alpha \left(i\dot{\varphi}(x)\right) + \delta_k^\alpha \int \mathrm{d}^3y \left([\varphi(x), \pi(y)]\partial^k_{(y)}\varphi(y) + \pi(y)\left[\varphi(x), \partial^k_{(y)}\varphi(y)\right]\right) \\
[\varphi(x), P^\alpha] &= \delta_0^\alpha \left(i\dot{\varphi}(x)\right) + \delta_k^\alpha \int \mathrm{d}^3y \left(i\delta^{(3)}(x-y)\partial^k_{(y)}\varphi(y)\right) \\
&\quad + \delta_k^\alpha \int \mathrm{d}^3y \left(\pi(y)\left[\varphi(x), \partial^k_{(y)}\right]\varphi(y) + \pi(y)\partial^k_{(y)}[\varphi(x), \varphi(y)]\right)
\end{aligned}$$

The commutator $[\varphi(x), \partial^k_{(y)}]$ vanishes because the derivatives are only with respect to y, and the commutator $[\varphi(x), \varphi(y)]$ vanishes by definition of the field commutation relations. This leaves

$$\begin{aligned}
[\varphi(x), P^\alpha] &= \delta_0^\alpha \left(i\partial^0\varphi(x)\right) + \delta_k^\alpha \left(i\partial^k\varphi(x)\right) \\
&\boxed{[\varphi(x), P^\alpha] = i\partial^\alpha\varphi(x)}
\end{aligned}$$

10.2

For the general Lagrangian

$$\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu\varphi_a, x^\mu)$$

under an active symmetry transformation $\varphi_a \to \varphi_a + \delta\varphi_a$, the variation in the Lagrangian is given by

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi_a}\delta\varphi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\delta(\partial_\mu\varphi_a) \\
&= \left(\frac{\partial\mathcal{L}}{\partial\varphi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\right)\right)\delta\varphi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\delta\varphi_a\right) \\
\delta\mathcal{L} &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}\delta\varphi_a\right)
\end{aligned}$$

if the Lagrangian satisfies the Euler-Lagrange equations of motion. The variation in the Lagrangian can equal a four-divergence without affecting the equations of motion

$$\delta\mathcal{L} = \partial_\mu W^\mu$$

Equating these, we have the conservation law $\partial_\mu J^\mu = 0$ where

$$\boxed{J^\mu = \sum_a \Pi^\mu_a \delta\varphi_a - W^\mu}$$

10.3

For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

the stress-energy tensor is given by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}$$
$$\boxed{T^{\mu\nu} = (\partial^\mu\varphi)(\partial^\nu\varphi) - \frac{1}{2}g^{\mu\nu}\left[(\partial_\mu\varphi)^2 - m^2\varphi^2\right]}$$

The T^{00} component is given by

$$T^{00} = \dot\varphi^2 - \frac{1}{2}\dot\varphi^2 + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2$$
$$\boxed{T^{00} = \frac{1}{2}\dot\varphi^2 + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 = \mathcal{H}}$$

which is exactly the Hamiltonian found in Exercise 6.1. Calculating the four-divergence of $T^{\mu\nu}$, we find

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial^2\varphi\partial^\nu\varphi + \partial^\mu\varphi(\partial_\mu\partial^\nu\varphi) - \frac{1}{2}g^{\mu\nu}\left[\partial_\mu(\partial_\lambda\varphi)(\partial^\lambda\varphi) - 2m^2\varphi(\partial_\mu\varphi)\right] \\
&= \partial^2\varphi\partial^\nu\varphi + \partial^\mu\varphi(\partial_\mu\partial^\nu\varphi) - \frac{1}{2}\partial^\nu(\partial_\lambda\varphi)(\partial^\lambda\varphi) + m^2\varphi(\partial^\nu\varphi) \\
&= (\partial^2 + m^2)\varphi(\partial^\nu\varphi) + \partial^\mu\varphi(\partial_\mu\partial^\nu\varphi) - \frac{1}{2}(\partial^\nu\partial_\lambda\varphi)(\partial^\lambda\varphi) - \frac{1}{2}(\partial_\lambda\varphi)(\partial^\nu\partial^\lambda\varphi) \\
&= \partial^\mu\varphi(\partial_\mu\partial^\nu\varphi) - \partial^\lambda\varphi(\partial_\lambda\partial^\nu\varphi)
\end{aligned}$$
$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

For the Noether charges, we have

$$\begin{aligned}
P^0 &= \int \mathrm{d}^3x\, T^{00} \\
&= \int \mathrm{d}^3x\, \mathcal{H}
\end{aligned}$$
$$\boxed{P^0 = H}$$

$$\begin{aligned}
P^i &= \int \mathrm{d}^3x\, T^{0i} \\
&= \int \mathrm{d}^3x\, \dot\varphi\partial^i\varphi
\end{aligned}$$
$$\boxed{P^i = \int \mathrm{d}^3x\, \pi\, \partial^i\varphi}$$

10.4

For the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

the conjugate momentum density is given by

$$\begin{aligned}
\Pi^{\sigma\rho} &= \frac{\partial \mathcal{L}}{\partial(\partial_\sigma A_\rho)} \\
&= \frac{\partial}{\partial(\partial_\sigma A_\rho)} \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \right] \\
&= -\frac{1}{2} (\delta^\sigma_\mu \delta^\rho_\nu - \partial^\sigma_\nu \partial^\rho_\mu) F^{\mu\nu} \\
&= -\frac{1}{2} (F^{\sigma\rho} - F^{\rho\sigma}) \\
&\boxed{\Pi^{\sigma\rho} = -F^{\sigma\rho}}
\end{aligned}$$

With this, the stress-energy tensor can be written as

$$\begin{aligned}
T^{\mu\nu} &= \Pi^{\mu\sigma} \partial^\nu A_\sigma - g^{\mu\nu} \mathcal{L} \\
&\boxed{T^{\mu\nu} = -F^{\mu\sigma} \partial^\nu A_\sigma + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}
\end{aligned}$$

The quantity $X^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$ is anti-symmetric in its first two indices since $F^{\mu\nu} = -F^{\nu\mu}$. Adding the four-divergence of this term to the stress-energy tensor, the conservation law is unchanged and the new tensor will have the form

$$\begin{aligned}
\tilde{T}^{\mu\nu} &= T^{\mu\nu} + \partial_\lambda X^{\lambda\mu\nu} \\
&= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + (\partial_\lambda F^{\mu\lambda}) A^\nu + F^{\mu\lambda} (\partial_\lambda A^\nu) \\
&= F^{\mu\lambda} (\partial_\lambda A^\nu - \partial^\nu A_\lambda) + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
&\boxed{\tilde{T}^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}
\end{aligned}$$

The $\tilde{T}^{00}$ element is given by

$$\begin{aligned}
\tilde{T}^{00} &= F^{0\lambda} F_{\lambda 0} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \\
&= \mathbf{E}^2 - \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) \\
&\boxed{\tilde{T}^{00} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2)}
\end{aligned}$$

and the $\tilde{T}^{0i}$ elements are given by

$$\begin{aligned}
\tilde{T}^{0i} &= F^{0\lambda} F_{\lambda i} \\
\tilde{T}^{01} &= E^2 B^3 - E^3 B^2 \\
\tilde{T}^{02} &= E^3 B^1 - E^1 B^3 \\
\tilde{T}^{03} &= E^1 B^2 - E^2 B^1 \\
&\boxed{\tilde{T}^{0i} = (\mathbf{E} \times \mathbf{B})^i}
\end{aligned}$$