Chapter 1

1.1

The path followed is the one which minimizes the total time traveled

\[
\tau = \int_{t_1}^{t_2} dt
\]
\[
= \frac{1}{c} \int_{t_1}^{t_2} c dt
\]
\[
= \frac{1}{c} \int_{t_1}^{t_2} \frac{c}{v} ds dt
\]
\[
= \frac{1}{c} \int_{A}^{B} n ds
\]
\[
\tau = \frac{n_1}{c} \sqrt{x^2 + y_1^2} + \frac{n_2}{c} \sqrt{(D - x)^2 + y_2^2}
\]
\[
\frac{\partial \tau}{\partial x} = 0
\]
\[
0 = n_1 \frac{x}{\sqrt{x^2 + y_1^2}} - n_2 \frac{D - x}{\sqrt{(D - x)^2 + y_2^2}}
\]
\[
0 = n_1 \sin \vartheta_1 - n_2 \sin \vartheta_2
\]

Therefore, the path followed is the one which satisfies

\[ n_1 \sin \vartheta_1 = n_2 \sin \vartheta_2 \]
The functional derivative of

\[ H[f] = \int G(x, y)f(y)\,dy \]

with respect to \( f(z) \) is given by

\[
\frac{\delta H[f]}{\delta f(z)} = \frac{d}{d\varepsilon} \left. H[f(y) + \varepsilon \delta(z - y)] \right|_{\varepsilon=0} = \int \frac{d}{d\varepsilon} \left[ G(x, y)(f(y) + \varepsilon \delta(z - y)) \right]_{\varepsilon=0} = \int G(x, y) \delta(z - y)\,dy
\]

The second functional derivative of

\[ I[f^3] = \int_{-1}^{1} f(x)^3\,dx \]

with respect to \( f(x_1) \) and \( f(x_0) \) is given by

\[
\frac{\delta^2 I[f^3]}{\delta f(x_1)\delta f(x_0)} = \frac{d}{d\varepsilon} \left[ 3(f(x_1) + \varepsilon \delta(x_0 - x_1))^2 \right]_{\varepsilon=0} = \int_{-1}^{1} 3f(x_1)^2 \delta(x_1 - x)\,dx = 3f(x_1)^2, \quad x_1 \in (-1, 1)
\]

The functional derivative of

\[ J[f] = \int \left( \frac{\partial f}{\partial y} \right)^2 \,dy \]

with respect to \( f(x) \) is given by

\[
\frac{\delta J[f]}{\delta f(x)} = \frac{d}{d\varepsilon} \left. J[f(y) + \varepsilon \delta(y - x)] \right|_{\varepsilon=0} = \int \frac{d}{d\varepsilon} \left[ \left( \frac{\partial f}{\partial y} + \varepsilon \frac{\partial}{\partial y} \delta(y - x) \right)^2 \right]_{\varepsilon=0} = \int 2 \frac{\partial f}{\partial y} \frac{\partial}{\partial y} \delta(y - x)\,dy = 2 \frac{\partial f}{\partial y} \delta(y - x) \bigg|_{\text{bound.}} = \int 2 \frac{\partial^2 f}{\partial y^2} \delta(y - x)\,dy
\]

The second functional derivative of

\[ J[f] = -2 \frac{\partial^2 f}{\partial x^2}, \quad x \in (\min(y), \max(y)) \]
The functional derivative of

$$G[f] = \int g(y, f) \, dy$$

with respect to $f(x)$ is given by

$$\frac{\delta G[f]}{\delta f(x)} = \frac{d}{dx} \int g(y, f(y) + \varepsilon \delta(x - y)) \, dy \bigg|_{\varepsilon=0}$$

$$= \int \frac{d}{dx} \left[ g(y, f) + \varepsilon \frac{\partial g}{\partial f} \delta(x - y) \right] \, dy \bigg|_{\varepsilon=0}$$

$$= \int \frac{\partial g}{\partial f} \delta(x - y) \, dy$$

The functional derivative of

$$H[f] = \int g(y, f, f') \, dy$$

where $f' = \frac{df}{dy}$, with respect to $f(x)$ is given by

$$\frac{\delta H[f]}{\delta f(x)} = \frac{d}{dx} \int g(y, f(y) + \varepsilon \delta(x - y), f'(y) + \varepsilon \frac{d}{dy} \delta(x - y)) \, dy \bigg|_{\varepsilon=0}$$

$$= \int \frac{d}{dx} \left[ g(y, f, f') + \varepsilon \frac{\partial g}{\partial f} \frac{d}{dy} \delta(x - y) + \varepsilon \frac{\partial g}{\partial f'} \frac{d^2}{dy^2} \delta(x - y) \right] \, dy \bigg|_{\varepsilon=0}$$

$$= \int \left( \frac{\partial g}{\partial f} \delta(x - y) + \frac{\partial g}{\partial f'} \frac{d}{dy} \delta(x - y) + \frac{\partial g}{\partial f''} \frac{d^2}{dy^2} \delta(x - y) \right) \, dy$$

$$= \frac{\partial g}{\partial f} + \frac{\partial g}{\partial f'} \frac{d}{dy} \delta(x - y) \bigg|_{\text{bound.}} - \int \frac{d}{dy} \left( \frac{\partial g}{\partial f'} \right) \frac{d}{dy} \delta(x - y) \, dy$$

The functional derivative of

$$J[f] = \int g(y, f, f', f'') \, dy$$

where $f'' = \frac{df'}{dy}$, with respect to $f(x)$ is given by

$$\frac{\delta J[f]}{\delta f(x)} = \frac{d}{dx} \int g(y, f + \varepsilon \delta(x - y), f' + \varepsilon \frac{d}{dy} \delta(x - y), f'' + \varepsilon \frac{d^2}{dy^2} \delta(x - y)) \, dy \bigg|_{\varepsilon=0}$$

$$= \int \frac{d}{dx} \left[ g(y, f, f', f'') + \varepsilon \frac{\partial g}{\partial f} \frac{d}{dy} \delta(x - y) + \varepsilon \frac{\partial g}{\partial f'} \frac{d^2}{dy^2} \delta(x - y) + \varepsilon \frac{\partial g}{\partial f''} \frac{d^3}{dy^3} \delta(x - y) \right] \, dy \bigg|_{\varepsilon=0}$$

$$= \int \left( \frac{\partial g}{\partial f} \frac{d}{dy} \delta(x - y) + \frac{\partial g}{\partial f'} \frac{d^2}{dy^2} \delta(x - y) + \frac{\partial g}{\partial f''} \frac{d^3}{dy^3} \delta(x - y) \right) \, dy$$

$$= \frac{\partial g}{\partial f} \frac{d}{dy} \delta(x - y) \bigg|_{\text{bound.}} - \int \frac{d^2}{dy^2} \left( \frac{\partial g}{\partial f'} \right) \frac{d}{dy} \delta(x - y) \, dy$$

$$= \frac{\partial g}{\partial f} \frac{d}{dy} \delta(x - y) + 0 - \frac{d}{dy} \left( \frac{\partial g}{\partial f''} \right) \frac{d^2}{dy^2} \delta(x - y) \bigg|_{\text{bound.}} + \int \frac{d^3}{dy^3} \left( \frac{\partial g}{\partial f''} \right) \delta(x - y) \, dy$$

$$= \frac{\partial g}{\partial f} \frac{d}{dy} \delta(x - y) + \frac{d^2}{dy^2} \left( \frac{\partial g}{\partial f'} \right) \delta(x - y) \bigg|_{\text{bound.}} + \int \frac{d^3}{dy^3} \left( \frac{\partial g}{\partial f''} \right) \delta(x - y) \, dy$$
CHAPTER 1.

1.4

The functional derivative of $\phi(x)$ with respect to $\phi(y)$ is given by

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \frac{d}{d\epsilon} \left[ \phi(x) + \epsilon \delta(x-y) \right]_{\epsilon=0} = \delta(x-y)$$

Similarly, the functional derivative of $\dot{\phi}(t)$ with respect to $\phi(t_0)$ is given by

$$\frac{\delta \dot{\phi}(t)}{\delta \phi(t_0)} = \frac{d}{d\epsilon} \left[ \dot{\phi}(t) + \epsilon \frac{d}{dt} \delta(t-t_0) \right]_{\epsilon=0} = \frac{d}{dt} \delta(t-t_0)$$

1.5

For a three-dimensional elastic medium with potential energy

$$V = \frac{T}{2} \int d^3x (\nabla \psi)^2$$

and kinetic energy

$$T = \frac{\rho}{2} \int d^3x \left( \frac{\partial \psi}{\partial t} \right)^2$$

the Lagrangian density is

$$\mathcal{L} = \frac{\rho}{2} \left( \frac{\partial \psi}{\partial t} \right)^2 - \frac{T}{2} (\nabla \psi)^2$$

With the action defined in terms of the Lagrangian density as

$$S[\psi] = \int d^4x \mathcal{L}(\psi, \partial_\mu \psi)$$

we obtain the equations of motion for $\psi$ by enforcing that the functional derivative of the action with respect to $\psi$ vanishes (see Exercise 1.3)

$$\frac{\delta S[\psi]}{\delta \psi} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) = 0$$

For this particular Lagrangian density, we have

$$-\frac{\partial}{\partial t} \left( \rho \frac{\partial \psi}{\partial t} \right) + \nabla \cdot (T \nabla \psi) = 0$$

$$-\rho \frac{\partial^2 \psi}{\partial t^2} + T \nabla^2 \psi = 0$$

$$\nabla^2 \psi = \frac{\rho}{T} \frac{\partial^2 \psi}{\partial t^2}$$

which is a wave equation with velocity $v = \sqrt{\frac{T}{\rho}}$. 
For a functional
\[ Z_0[J] = \exp \left( -\frac{1}{2} \int d^4x \; d^4y \; J(x) \Delta(x - y)J(y) \right) \]

where \( \Delta(x) = \Delta(-x) \). The functional derivative of \( Z_0 \) with respect to \( J(z_1) \) is given by

\[
\frac{\delta Z_0[J]}{\delta J(z_1)} = \frac{d}{d\varepsilon} \exp \left[ -\frac{1}{2} \int d^4x \; d^4y \; \left( J(x) + \varepsilon\delta(x - z_1) \right) \Delta(x - y) \left( J(y) + \varepsilon\delta(y - z_1) \right) \right] \bigg|_{\varepsilon=0} \\
= \left( -\frac{1}{2} \int d^4x \; d^4y \; \left( J(x)\delta(y - z_1)\Delta(x - y) + J(y)\delta(x - z_1)\Delta(x - y) \right) \right) Z_0[J] \\
= -\frac{1}{2} \left( \int d^4x \; J(x)\Delta(x - z_1) + \int d^4y \; J(y)\Delta(z_1 - y) \right) Z_0[J] \\
\frac{\delta Z_0[J]}{\delta J(z_1)} = - \left[ \int d^4y \; \Delta(z_1 - y)J(y) \right] Z_0[J]