Part I - Motivation and Foundation

2.1) $\langle q_F | e^{-iHT} | q_I \rangle = \left(\prod_{j=1}^{N-1} \int dq_j \right) \langle q_F | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\delta t} | q_{N-2} \rangle \dots \langle q_1 | e^{-iH\delta t} | q_I \rangle$

$$H = \frac{p^2}{2m} + V(q)$$

$$\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle = \int \frac{dp}{2\pi} \langle q_{j+1} | p \rangle \langle p | e^{-i\delta t (p^2/2m + V(q))} | q_j \rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\delta t (p^2/2m + V(q_j))} e^{ip(q_{j+1} - q_j)}$$

$$= \left(\frac{-im}{2\pi \delta t} \right)^{1/2} e^{im(q_{j+1} - q_j)^2/2\delta t - i\delta t V(q_j)}$$

$$\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle = \left(\frac{-im}{2\pi \delta t} \right)^{1/2} e^{i\delta t (m/2)[(q_{j+1} - q_j)/\delta t]^2 - i\delta t V(q_j)}$$

$$\langle q_F | e^{-iHT} | q_I \rangle = \left(\frac{-im}{2\pi \delta t} \right)^{N/2} \left(\prod_{k=1}^{N-1} \int dq_k \right) e^{i\delta t \sum_{j=0}^{N-1} (m/2)[(q_{j+1} - q_j)/\delta t]^2 - V(q_j)}$$

$$\boxed{\langle q_F | e^{-iHT} | q_I \rangle \to \int \mathcal{D}q(t) \exp\left\{ i \int_0^T dt \left(\tfrac{1}{2} m \dot{q}^2 - V(q) \right) \right\}}$$

2.2) $\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_N \, e^{-\frac{1}{2} x_i A_{ij} x_j + J_i x_i} = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} \equiv I$

$$\frac{\partial I}{\partial J_k} = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} \left(\tfrac{1}{2} A^{-1}_{kj} J_j + \tfrac{1}{2} J_i A^{-1}_{ik} \right) e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$= \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} A^{-1}_{kj} J_j \, e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$\left. \frac{\partial I}{\partial J_k} \right|_{J=0} = 0$$

$$\langle x_k \rangle = 0$$

$$\frac{\partial}{\partial J_\ell} \left(\frac{\partial I}{\partial J_k} \right) = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} \left(A^{-1}_{k\ell} + A^{-1}_{kj} J_j A^{-1}_{\ell i} J_i \right) e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$\left. \frac{\partial}{\partial J_\ell} \left(\frac{\partial I}{\partial J_k} \right) \right|_{J=0} = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} A^{-1}_{k\ell}$$

$$\langle x_k x_\ell \rangle = A^{-1}_{k\ell}$$

$$\frac{\partial}{\partial J_m} \left(\frac{\partial}{\partial J_\ell} \left(\frac{\partial I}{\partial J_k} \right) \right) = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} \Big[\left(A^{-1}_{km} A^{-1}_{\ell i} J_i + A^{-1}_{kj} J_j A^{-1}_{\ell m} \right) e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$
$$+ \left(A^{-1}_{k\ell} + A^{-1}_{kj} J_j A^{-1}_{\ell i} J_i \right) A^{-1}_{ma} J_a \, e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} \Big]$$
$$= \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} \left(A^{-1}_{km} A^{-1}_{\ell i} J_i + A^{-1}_{kj} J_j A^{-1}_{\ell m} + A^{-1}_{k\ell} A^{-1}_{ma} J_a + A^{-1}_{kj} J_j A^{-1}_{\ell i} J_i A^{-1}_{ma} J_a \right) e^{\frac{1}{2} J_i A^{-1}_{ij} J_j}$$

$$\left. \frac{\partial}{\partial J_m} \left(\frac{\partial}{\partial J_\ell} \left(\frac{\partial I}{\partial J_k} \right) \right) \right|_{J=0} = 0$$

$$\langle x_k x_\ell x_m \rangle = 0$$

$$\partial_n \partial_m \partial_\ell \partial_k I = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} \Big[\big(A^{-1}_{km} A^{-1}_{\ell n} + A^{-1}_{kn} A^{-1}_{\ell m} + A^{-1}_{k\ell} A^{-1}_{mn} + A^{-1}_{kn} A^{-1}_{\ell i} J_i A^{-1}_{ma} J_a + A^{-1}_{kj} J_j A^{-1}_{\ell n} A^{-1}_{ma} J_a$$
$$+ A^{-1}_{kj} J_j A^{-1}_{\ell i} J_i A^{-1}_{mn} \big) e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} + \big(A^{-1}_{km} A^{-1}_{\ell i} J_i + A^{-1}_{kj} J_j A^{-1}_{\ell m} + A^{-1}_{k\ell} A^{-1}_{ma} J_a$$
$$+ A^{-1}_{kj} J_j A^{-1}_{\ell i} J_i A^{-1}_{ma} J_a \big) A^{-1}_{nb} J_b \, e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} \Big]$$

$$\left. \partial_n \partial_m \partial_\ell \partial_k I \right|_{J=0} = \left(\frac{(2\pi)^N}{\det(A)} \right)^{1/2} \left(A^{-1}_{km} A^{-1}_{\ell n} + A^{-1}_{kn} A^{-1}_{\ell m} + A^{-1}_{k\ell} A^{-1}_{mn} \right)$$

$$\langle x_k x_\ell x_m x_n \rangle = A^{-1}_{k\ell} A^{-1}_{mn} + A^{-1}_{km} A^{-1}_{\ell n} + A^{-1}_{kn} A^{-1}_{\ell m}$$

$$\boxed{\langle x_i x_j \dots x_m x_n \rangle = \sum_{\{i,j,\dots,m,n\}} (A^{-1}_{ij}) \dots (A^{-1}_{mn})}$$