Problem 6.7

For a potential

\[ V = \begin{cases} 
0 & r < a \\
\infty & r > a 
\end{cases} \]  

we can use the same method as in problem 6.4 to find the s-wave phase shift and the total cross section.

a) For \( r > a \), we know the solution to the radial Schrödinger equation will have the form

\[ A_{\ell}(r) = e^{i\delta_{\ell}} \left( \cos(\delta_{\ell}) j_{\ell}(kr) - \sin(\delta_{\ell}) n_{\ell}(kr) \right) \]  

However, for \( r < a \), the wave function must vanish for all \( \ell \)

\[ A_{\ell}(r) = 0 \quad \forall \ell \]  

Therefore, we need only to ensure the continuity of the wave function at \( r = a \). Namely,

\[ e^{i\delta_{\ell}} \left( \cos(\delta_{\ell}) j_{\ell}(kr) - \sin(\delta_{\ell}) n_{\ell}(kr) \right) = 0 \]  

This leads to

\[ \tan \delta_{\ell} = \frac{j_{\ell}(ka)}{n_{\ell}(ka)} \]  

Using this, the s-wave phase shift is obtained immediately

\[ \tan \delta_{0} = \frac{j_{0}(ka)}{n_{0}(ka)} \]  

\[ = - \tan(ka) \]  

\[ \therefore \delta_{0} = -ka \]  

b) The differential cross-section is given by

\[ \frac{d\sigma}{d\Omega} = \frac{1}{k} \left| \sum_{\ell} (2\ell + 1) e^{i\delta_{\ell}} \sin \delta_{\ell} P_{\ell}(\cos \theta) \right|^2 \]  

\[ \approx \frac{1}{k} \sin \delta_{0}^2 \]  

\[ = \frac{\sin^2 \delta_{0}}{k^2} \]  

\[ \approx \left( \frac{\delta_{0}}{k} \right)^2 \]  

\[ \frac{d\sigma}{d\Omega} = a^2 \]  

for \( ka \ll 1 \). Integrating this over all angular dependence, we get

\[ \sigma_{\text{tot}} = 4\pi a^2 \]  

which is different by a factor of 4 from the geometric cross-section of a hard sphere of radius \( a \). Interestingly enough, this is exactly the surface area of a hard sphere.