Problem 6.4

a) For the potential

\[ V(r) = \begin{cases} V_0 & r < R \\ 0 & r > R \end{cases} \]  

we can use the method of partial waves to calculate the differential cross section in the limit that \(|V_0| \ll E = \hbar^2 k^2 / 2m\) and \(kR \ll 1\). Following the procedure laid out in Sakurai, for \(r > R\), we have

\[ A^\text{out}_\ell = e^{i\delta_\ell} \left( \cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) n_\ell(kr) \right) \]  

The logarithmic derivative of this wave function yields the following equation for \(\delta_\ell\)

\[ \tan(\delta_\ell) = \frac{kR j'_\ell(kR) - \beta_\ell j_\ell(kR)}{kR n'_\ell(kR) - \beta_\ell n_\ell(kR)} \]  

As for \(r < R\), the potential is constant, so our solution to the radial equation is simply spherical Bessel functions of the first kind (since the wave function cannot be singular at the origin)

\[ A^\text{in}_\ell = j_\ell(\kappa r) \]  

where

\[ \kappa^2 = \frac{2m}{\hbar^2} (E - V_0) \]  

Therefore, we can calculate

\[ \beta_\ell = \kappa R \frac{j'_\ell(\kappa R)}{j_\ell(\kappa R)} \]  

In order to deal with the derivatives, we can make use of the following relationship

\[ f'_\ell(x) = \ell f_\ell(x) - f_{\ell+1}(x) \]  

Using this, we have

\[ \beta_\ell = \ell - \kappa R \frac{j_{\ell+1}(\kappa R)}{j_\ell(\kappa R)} \]  

\[ kR j'_\ell(kR) - \beta_\ell j_\ell(kR) = \kappa R \frac{j_{\ell+1}(\kappa R)}{j_\ell(\kappa R)} f_\ell(kR) - kR f_{\ell+1}(kR) \]  

In the limit that \(kR \ll 1\) and \(\kappa R \ll 1\), we can approximate the spherical Bessel functions as

\[ j_\ell(x) \approx \frac{x^\ell}{(2\ell + 1)!!}, \quad n_\ell(x) \approx -\frac{(2\ell - 1)!!}{x^{\ell+1}} \]  

Therefore,

\[ \kappa R \frac{j_{\ell+1}(\kappa R)}{j_\ell(\kappa R)} \approx \frac{(\kappa R)^2}{2\ell + 3} \]  

Using this, the numerator of eq. 7 becomes

\[ \frac{(\kappa R)^2}{(2\ell + 3)} j_\ell(kR) - kR j_{\ell+1}(kR) = \frac{(\kappa R)^2}{(2\ell + 3)} \frac{(kR)^\ell}{(2\ell + 1)!!} - \frac{(kR)^{\ell+2}}{(2\ell + 3)!!} \]  

\[ = \frac{(kR)^\ell}{(2\ell + 3)!!} [(kR)^2 - (\kappa R)^2] \]  

while the denominator becomes

\[ \frac{(\kappa R)^2}{(2\ell + 3)} j_\ell(kR) - kR j_{\ell+1}(kR) = \frac{(\kappa R)^2}{(2\ell + 3)} \frac{(kR)^\ell}{(2\ell + 1)!!} - \frac{(kR)^{\ell+2}}{(2\ell + 3)!!} \]  

\[ = \frac{(kR)^\ell}{(2\ell + 3)!!} [(kR)^2 - (\kappa R)^2] \]
\[
\frac{(\kappa R)^2}{(2\ell + 3)} n_\ell(kR) - kR n_{\ell+1}(kR) = - \frac{(\kappa R)^2}{(2\ell + 3)} \frac{(2\ell + 1)!!}{(kR)^{\ell+1}} + kR \frac{(2\ell + 1)!!}{(kR)^{\ell+2}}
\]

\[
= \frac{(2\ell + 1)!!}{(kR)^{\ell+1}} \left[ 1 - \frac{(\kappa R)^2}{2\ell + 3} \right]
\]

\[
\approx \frac{(2\ell + 1)!!}{(kR)^{\ell+1}}.
\]

Combining the two, we find

\[
\tan(\delta_\ell) \approx \frac{(kR)^{2\ell+3}}{(2\ell + 1)!!(2\ell + 3)!!} \left( 1 - \frac{\kappa^2}{k^2} \right)
\]

For \( \ell = 0 \), we have

\[
\tan(\delta_0) = \frac{(kR)^3}{3} \left[ 1 - \frac{E - V_0}{E} \right]
\]

\[
= \frac{(kR)^3}{3} \left( \frac{V_0}{E} \right)
\]

\[
= \frac{2mkR^3V_0}{3h^2}
\]

Since \(|V_0| \ll E\), this yields

\[
\tan(\delta_0) \approx \delta_0 = \frac{2mkR^3V_0}{3h^2}
\]

Therefore,

\[
\frac{d\sigma}{d\Omega} \approx \frac{1}{k^2} \left( \frac{2mR^3V_0k}{3h^2} \right)^2
\]

the differential cross-section can be seen to be isotropic. Calculating the total cross-section, we find

\[
\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega
\]

\[
= 4\pi \frac{2mR^3V_0k}{3h^2}^2
\]

\[
\sigma_{\text{tot}} = \frac{16\pi m^2 V_0^2 R^6}{9h^4}
\]

b) If the energy is raised slightly, we’ll consider the contribution of the p-wave (\( \ell = 1 \)).

\[
\tan(\delta_1) = \frac{(kR)^5}{45} \left( \frac{V_0}{E} \right)
\]

\[
= \frac{(kR)^5}{45} \frac{2mV_0}{h^2k^2}
\]

\[
\delta_1 \approx \frac{2mR^5V_0k^3}{45h^2}
\]

With this, the differential cross section is

\[
\frac{d\sigma}{d\Omega} \approx \left| \frac{\delta_0}{k} e^{i\delta_0} + \frac{3\delta_1}{k} e^{i\delta_1} \cos \theta \right|^2
\]
Neglecting terms of $O(\delta_1^2)$, this has the form

$$\frac{d\sigma}{d\Omega} = A + B \cos \theta$$ \hspace{1cm} (34)

where

$$A = \frac{\delta_0^2}{k^2}, \quad B = \frac{6\delta_0 \delta_1}{k^2} \cos(\delta_0 - \delta_1) \approx \frac{6\delta_0 \delta_1}{k^2}$$ \hspace{1cm} (35)

Thus,

$$\frac{B}{A} = \frac{6\delta_1}{\delta_0} = \frac{2}{5} (kR)^2$$ \hspace{1cm} (36)