Problem 5.9

a) For a \( p \)-orbital electron in the state \(|n, \ell, m = 0, \pm 1\rangle\) subject to a potential of the form \( V = \lambda(x^2 - y^2) \), the zeroth-order energy eigenstates are obtained by diagonalizing the perturbing potential in the degenerate subspace. The matrix elements will be of the form

\[
V_{m'0} = \langle n, \ell, m' | (x^2 - y^2) | n, \ell, m \rangle
\]

We can express the quantity \( x^2 - y^2 \) as the sum of two rank-two spherical tensors

\[
V_{m'0} = \frac{\lambda}{2} \langle n, \ell, m' | (T^{(2)}_{1,1} + T^{(2)}_{1,-1}) | n, \ell, m \rangle
\]

In this form, the Wigner-Eckart theorem tells us that we must have

\[
m' = m \pm 2
\]

for any non-vanishing matrix elements. Therefore, we have

\[
V \to \begin{pmatrix}
0 & 0 & I \\
0 & 0 & 0 \\
I & 0 & 0
\end{pmatrix}
\]

where \( I \) represents the non-vanishing product of the Clebsch-Gordon coefficient and the rank-two reduced matrix element specified by the Wigner-Eckart theorem. We can now diagonalize the perturbation, which yields the first-order shift to the energy eigenvalues

\[
\begin{vmatrix}
-\Delta & 0 & I \\
0 & -\Delta & 0 \\
I & 0 & -\Delta
\end{vmatrix} = 0
\]

\[
\Delta = 0, \pm I
\]

Since we have three distinct shifts, it’s clear that the three-fold degeneracy is lifted. We can show this explicitly by calculating the zeroth-order energy eigenkets

<table>
<thead>
<tr>
<th>Zeroth-Order State</th>
<th>First-Order Energy Shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{\sqrt{2}}</td>
<td>n, 1, 1\rangle - \frac{1}{\sqrt{2}}</td>
</tr>
<tr>
<td>(</td>
<td>n, 1, 0\rangle )</td>
</tr>
<tr>
<td>( \frac{1}{\sqrt{2}}</td>
<td>n, 1, 1\rangle + \frac{1}{\sqrt{2}}</td>
</tr>
</tbody>
</table>

b) The spherical representation of the state \(|n, \ell, m\rangle\) in the position basis is given by

\[
\langle \mathbf{x}|n, \ell, m\rangle = R_{n\ell}(r)Y^m_{\ell}(\theta, \phi)
\]

Using the anti-unitary property of the time reversal operator,

\[
\langle \mathbf{x}|\Theta\alpha\rangle = \langle \mathbf{x}|\alpha\rangle^*,
\]

we find that

\[
\langle \mathbf{n}|\Theta|n, \ell, m\rangle = \langle \mathbf{n}|\Theta|n, \ell, m\rangle^* = \left(R_{n\ell}(r)Y^m_{\ell}(\theta, \phi)\right)^* = (-1)^m R_{n\ell}(r)Y^{m*}_{\ell}(\theta, \phi).
\]

With this, we can show that indeed the states calculated above in part (a) go into themselves under time reversal (up to an overall phase)
\[ \Theta \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle - \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) \rightarrow \Theta \frac{1}{\sqrt{2}} \left( R_{n,1}(r)Y_1^0(\theta, \phi) - R_{n,1}(r)Y_1^{-1}(\theta, \phi) \right) \]

\[ \rightarrow \frac{1}{\sqrt{2}} \left( (-1)^j R_{n,1}(r)Y_1^{-1}(\theta, \phi) - (-1)^{-1} R_{n,1}(r)Y_1^{1}(\theta, \phi) \right) \quad (64) \]

\[ \rightarrow \frac{1}{\sqrt{2}} \left( R_{n,1}(r)Y_1^{1}(\theta, \phi) - R_{n,1}(r)Y_1^{-1}(\theta, \phi) \right) \quad (65) \]

\[ \Theta \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle - \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) = \frac{1}{\sqrt{2}}|n, 1, 1\rangle - \frac{1}{\sqrt{2}}|n, 1, -1\rangle \quad (66) \]

\[ \Theta |n, 1, 0\rangle \rightarrow \Theta \left( R_{n,1}Y_1^0(\theta, \phi) \right) \]

\[ \rightarrow (-1)^j R_{n,1}Y_1^{0}(\theta, \phi) \quad (68) \]

\[ \rightarrow R_{n,1}(r)Y_1^{0}(\theta, \phi) \quad (69) \]

\[ \Theta |n, 1, 0\rangle = |n, 1, 0\rangle \quad (70) \]

\[ \Theta \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle + \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) \rightarrow \Theta \frac{1}{\sqrt{2}} \left( R_{n,1}(r)Y_1^0(\theta, \phi) + R_{n,1}(r)Y_1^{-1}(\theta, \phi) \right) \]

\[ \rightarrow \frac{1}{\sqrt{2}} \left( (-1)^j R_{n,1}(r)Y_1^{-1}(\theta, \phi) + (-1)^{-1} R_{n,1}(r)Y_1^{1}(\theta, \phi) \right) \quad (72) \]

\[ \rightarrow -\frac{1}{\sqrt{2}} \left( R_{n,1}(r)Y_1^{1}(\theta, \phi) + R_{n,1}(r)Y_1^{-1}(\theta, \phi) \right) \quad (73) \]

\[ \Theta \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle + \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) = e^{i\pi} \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle + \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) \quad (74) \]

\[ \Theta \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle + \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) = e^{i\pi} \left( \frac{1}{\sqrt{2}}|n, 1, 1\rangle + \frac{1}{\sqrt{2}}|n, 1, -1\rangle \right) \quad (75) \]