Problem 5.23

For the force $F(t) = F_0 e^{-t/\tau}$, we can associate a potential in one-dimension

$$V(x, t) = -\int_0^x F(t) \, dx' = -F_0 x e^{-t/\tau}$$  \hspace{1cm} (1)

a) Using time-dependent perturbation theory, we can calculate the probability of the ground state undergoing a transition to another state

$$c_n^{(1)}(t) = -\frac{i}{\hbar} \int_0^t V_{n0}(x, t') \, e^{i \omega_n t'} \, dt', \quad n > 0$$  \hspace{1cm} (2)

First, we have that

$$\omega_{n0} = \frac{E_n - E_0}{\hbar} = \frac{1}{\hbar} \left[ \hbar \omega \left( n + \frac{1}{2} \right) - \frac{1}{2} \hbar \omega \right] = n\omega$$  \hspace{1cm} (3)

Second, we evaluate the matrix element $V_{n0}(x, t)$

$$V_{n0}(x, t) = \langle n| V(x, t) |0 \rangle$$  \hspace{1cm} (4)

$$= -F_0 e^{-t/\tau} \langle n|x|0 \rangle$$  \hspace{1cm} (5)

$$= -F_0 e^{-t/\tau} \sqrt{\frac{\hbar}{2m\omega}} \langle n|(a + a^\dagger)|0 \rangle$$  \hspace{1cm} (6)

$$V_{n0}(x, t) = -F_0 \sqrt{\frac{\hbar}{2m\omega}} \delta_{n1} e^{-t/\tau}$$  \hspace{1cm} (7)

With this, we have

$$c_n^{(1)}(t) = \frac{i F_0}{\sqrt{2\hbar m\omega}} \delta_{n1} \int_0^t e^{(i \omega_n - 1/\tau)t'} \, dt'$$  \hspace{1cm} (8)

$$= \frac{i F_0}{\sqrt{2\hbar m\omega}} \delta_{n1} e^{(i \omega_n - 1/\tau)t} \bigg|_0$$  \hspace{1cm} (9)

$$= \frac{i F_0}{\sqrt{2\hbar m\omega}} (i \omega - 1/\tau) \left( e^{(i \omega - 1/\tau)t} - 1 \right)$$  \hspace{1cm} (10)

For $n = 1$, this yields

$$c_1^{(1)}(t) = \frac{i F_0}{\sqrt{2\hbar m\omega}} (i \omega - 1/\tau) \left( e^{(i \omega - 1/\tau)t} - 1 \right)$$  \hspace{1cm} (11)

with probability

$$|c_1^{(1)}(t)|^2 = \frac{1}{2 \hbar m\omega \omega^2 + 1/\tau^2} \left[ e^{-2t/\tau} - 2 e^{-t/\tau} \cos(\omega t) + 1 \right]$$  \hspace{1cm} (12)

In the limit as $t \to \infty$, this probability becomes independent of time

$$\lim_{t \to \infty} |c_1^{(1)}(t)|^2 = \frac{1}{2 \hbar m\omega \omega^2 + 1/\tau^2} = |c_1^{(1)}|^2$$  \hspace{1cm} (13)

This makes sense, since the perturbing potential goes to zero as $t \to \infty$, and so the perturbed eigenkets will become constant in time with a constant, non-zero probability of being measured in the first-excited state.

b) At first order, no we cannot find any higher states. This is due to the kronecker delta present in the matrix element calculated in equation 7
However, at second order, we find that

\[ |c_n^{(1)}(t)|^2 = 0, \quad n \geq 2 \]  

(14)

Thus, we can conclude that in order to calculate a transition probability from the ground state to some \(n\)th state, we require \(n\)th-order time-dependent perturbation theory.