Problem 4.7

a) Let $\psi(x, t)$ represent the wave function of a spinless particle in three dimensions. The time reversal operator ($\Theta$) by definition takes $t \to -t$, and, as an anti-unitary operator, also takes $c \to c^*$. Therefore, the action of the time reversal operator on the wave function will produce

$$\psi(x, t) \xrightarrow{\Theta} \psi^*(x, -t). \quad (1)$$

However, since this wave function corresponds to plane waves, we know it must be an eigenket of the momentum operator (expressed in the position basis)

$$\psi(x, t) \to |p(t)\rangle = U|p\rangle, \quad (2)$$

where $U$ is the standard, unitary, time evolution operator. If we want to consider the action of the time reversal operator on this state, we need to know how the two time operators interact. Using the fact that $[\Theta, H] = 0$, it can be shown that

$$\Theta U \Theta^{-1} = U^\dagger \Rightarrow \Theta U = U^\dagger \Theta. \quad (3)$$

Because the momentum operator is odd under time reversal symmetry, we also know that

$$\Theta |p\rangle = |-p\rangle. \quad (4)$$

Therefore, we can conclude that

$$\psi(x, t) \xrightarrow{\Theta} \Theta U |p\rangle = U^\dagger \Theta |p\rangle = U^\dagger |-p\rangle = |-p(-t)\rangle. \quad (5)$$

Comparing equations 1 and 5, we find that the state $\psi^*(x, -t)$ corresponds to a momentum eigenstate (i.e. a plane wave) with the direction of the momentum reversed.

b) Let $\chi(\hat{n})$ represent the two-component eigenspinor of $\sigma \cdot \hat{n}$ with eigenvalue $+1$. In terms of the polar and azimuthal angles, $\beta$ and $\gamma$ respectively, this spinor can be represented as

$$\chi(\beta, \gamma) \to \left( \begin{array}{c} \cos \left( \frac{\beta}{2} \right) e^{-i\gamma/2} \\ \sin \left( \frac{\beta}{2} \right) e^{i\gamma/2} \end{array} \right). \quad (6)$$

When the time reversal operator acts on a spin $\frac{1}{2}$ state, it serves to reverse the spin of the state, which in this case would yield the eigenspinor of $\sigma \cdot \hat{n}$ with eigenvalue $-1$. Defining this complementary eigenspinor as $\varphi(\beta, \gamma)$, we can write

$$\chi(\beta, \gamma) \xrightarrow{\Theta} \varphi(\beta, \gamma) \to \left( \begin{array}{c} -\sin \left( \frac{\beta}{2} \right) e^{-i\gamma/2} \\ \cos \left( \frac{\beta}{2} \right) e^{i\gamma/2} \end{array} \right). \quad (7)$$

Now, if we calculate the quantity $-i\sigma_2\chi(\hat{n})$.

$$-i\sigma_2\chi(\hat{n}) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} \cos \left( \frac{\beta}{2} \right) e^{i\gamma/2} \\ \sin \left( \frac{\beta}{2} \right) e^{-i\gamma/2} \end{array} \right) = \left( \begin{array}{c} -\sin \left( \frac{\beta}{2} \right) e^{-i\gamma/2} \\ \cos \left( \frac{\beta}{2} \right) e^{i\gamma/2} \end{array} \right). \quad (8)$$

we find that this is precisely equal to $\varphi(\beta, \gamma)$, which, by construction, is the two-component eigenspinor $\chi(\beta, \gamma)$ with the spin reversed.