Problem 2.32

Using the definition
\[ K(x', t; x', 0) = \sum_{a'} |\langle x'|a' \rangle|^2 e^{-iE_{a't}/\hbar} \tag{79} \]
we can see that
\[ Z = \int K(x', t; x', 0) d^3x' = \sum_{a'} e^{-iE_{a't}/\hbar} = \sum_{a'} e^{-E_{a't}} \tag{80} \]

Using this, the quantity in question is
\[ -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = \sum_{a'} \frac{E_{a't} \exp [-E_{a't}]}{\exp [-E_{a't}]} \tag{81} \]

When we take the limit of \( \beta \to \infty \), the term with the lowest value of \( E_{a't} \) will dominate (i.e. the ground state). Denoting this state as having energy \( E_0 \), we have
\[ \lim_{\beta \to \infty} \left( -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = E_0 \exp [-E_0] \exp [-E_0] \tag{82} \]
\[ \lim_{\beta \to \infty} \left( -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = E_0 \tag{83} \]

In the case of a particle of mass \( m \) in a box of length \( L \), the energy levels are given by
\[ E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \tag{84} \]

Using the equations above, we have
\[ -\frac{1}{Z} \frac{\partial Z}{\partial \beta} = \frac{\hbar^2 \pi^2}{2mL^2} \sum_n \frac{n^2 \exp [-E_n]}{\exp [-E_n]} \tag{85} \]
\[ \lim_{\beta \to \infty} \left( -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = \frac{\hbar^2 \pi^2}{2mL^2} \frac{1^2 \exp [-E_1]}{\exp [-E_1]} \tag{86} \]
\[ \lim_{\beta \to \infty} \left( -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = \frac{\hbar^2 \pi^2}{2mL^2} \tag{87} \]

which yields the ground state energy.