Problem 2.28

a) The energy eigenfunctions of this system are the solutions to the Schrödinger equation in cylindrical coordinates in the absence of a potential

\[
-\frac{\hbar^2}{2m_e} \nabla^2 \psi = E \psi \quad \Rightarrow \quad \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial z^2} = -k^2 \psi
\]  

(38)

If we assume a solution of the form \( \psi(\rho, \theta, z) = R(\rho) \Theta(\theta) Z(z) \), we obtain

\[
\frac{1}{\rho} \Theta Z \left( \frac{dR}{d\rho} + \frac{\rho^2 R}{\rho^2} \frac{d^2 R}{d\rho^2} \right) + \frac{1}{\rho^2} R Z \frac{d^2 \Theta}{d\theta^2} + R \Theta \frac{d^2 Z}{dz^2} = -k^2 R \Theta Z
\]  

(39)

Dividing through by \( \frac{1}{\rho^2} R \Theta Z \), this becomes

\[
\frac{\rho}{R} \left( \frac{dR}{d\rho} + \frac{\rho^2 R}{\rho^2} \frac{d^2 R}{d\rho^2} \right) + \frac{1}{\rho^2} \frac{d^2 \Theta}{d\theta^2} + \frac{\rho^2 d^2 Z}{Z \, dz^2} + \rho^2 k^2 = 0
\]  

(40)

Since the \( \theta \) dependence is isolated in one term, we can write

\[
\frac{1}{\rho^2} \frac{d^2 \Theta}{d\theta^2} = -m^2
\]  

(41)

\[
\frac{\rho}{R} \left( \frac{dR}{d\rho} + \frac{\rho^2 R}{\rho^2} \frac{d^2 R}{d\rho^2} \right) + \frac{\rho^2 d^2 Z}{Z \, dz^2} + \rho^2 k^2 = m^2
\]  

(42)

Equation 41 yields solutions of the form

\[
\Theta(\theta) = e^{\pm im\theta}
\]  

(43)

From the periodic boundary condition \( \Theta(\theta) = \Theta(\theta + 2\pi) \), we can see that \( m \) is a positive, real integer. As for equation 42, we can now write

\[
\frac{1}{R} \left( \frac{d^2 R}{d\rho^2} + \frac{dR}{\rho \, d\rho} \right) + \left( \frac{k^2 - \frac{m^2}{\rho^2}}{\rho^2} \right) + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0
\]  

(44)

Since the \( z \) dependence is isolated in one term, we can write

\[
\frac{1}{Z} \frac{d^2 Z}{dz^2} = -\alpha^2
\]  

(45)

\[
\frac{1}{R} \left( \frac{d^2 R}{d\rho^2} + \frac{dR}{\rho \, d\rho} \right) + \left( \frac{k^2 - \frac{m^2}{\rho^2}}{\rho^2} \right) = \alpha^2
\]  

(46)

Equation 45 yields solutions of the form

\[
Z(z) = c_1 e^{i\alpha z} + c_1 e^{-i\alpha z}
\]  

(47)

Applying the boundary conditions \( Z(0) = Z(L) = 0 \), we obtain

\[
Z(0) = 0 \quad \Rightarrow \quad c_2 = -c_1
\]  

(48)

\[
Z(L) = 0 \quad \Rightarrow \quad \alpha L = \ell \pi, \quad \ell \in \mathbb{Z}
\]  

(49)
Therefore, equation 47 becomes
\[ Z(z) = C \sin \left( \frac{\ell \pi z}{L} \right) \] (50)

Lastly, we have equation 46 to solve. We can rewrite it in the form
\[ \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left( k^2 - \alpha^2 - \frac{m^2}{\rho^2} \right) R = 0 \] (51)

If we define \( \kappa^2 \equiv k^2 - \alpha^2 \) and \( x \equiv \kappa \rho \), this becomes
\[ x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + (x^2 - m^2) R = 0 \] (52)

This is the Bessel differential equation, and yields Bessel functions as the solution
\[ R(\rho) = A_m J_m(\kappa \rho) + B_m N_m(\kappa \rho) \] (53)

Applying the boundary conditions \( R(\rho_a) = R(\rho_b) = 0 \) yields the following system of equations
\[
\begin{pmatrix}
J_m(\kappa \rho_a) & N_m(\kappa \rho_a) \\
J_m(\kappa \rho_b) & N_m(\kappa \rho_b)
\end{pmatrix}
\begin{pmatrix}
A_m \\
B_m
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\] (54)

The \( n \) values of \( \kappa, \) for a given value of \( m, \) which permit non-trivial solutions to this system are given by the \( n \) roots of the determinant of the matrix in the above equation. If we label the \( n \)th root of the determinant as \( k_{mn}, \) then the characteristic equation reads
\[ J_m(k_{mn}\rho_a)N_m(k_{mn}\rho_b) - J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) = 0 \] (55)

Therefore, the energy eigenfunctions are given by
\[ \psi_{\ell mn}(\rho, \theta, z) = R_{\ell mn}(\rho)\Theta_m(\theta)Z_{\ell}(z) \] (56)
\[ \psi_{\ell mn}(\rho, \theta, z) = C \left( A_m J_m(k_{mn}\rho) + B_m N_m(k_{mn}\rho) \right) e^{im\theta} \sin \left( \frac{\ell \pi z}{L} \right) \] (57)

with energy eigenvalues
\[ E = \hbar^2 k^2 \frac{2m_e}{2m_e} = \hbar^2 \left( \frac{\kappa^2 + \alpha^2}{2m_e} \right) \] (58)
\[ E = \hbar^2 \left[ k_{mn}^2 + \left( \frac{\ell \pi}{L} \right)^2 \right] \] (59)

b) In the presence of a magnetic field, \( \mathbf{B} = B\hat{z}, \) the Hamiltonian of the system becomes
\[ H = \frac{1}{2m_e} \left( \mathbf{p} - \frac{e\mathbf{A}}{c} \right) \cdot \left( \mathbf{p} - \frac{e\mathbf{A}}{c} \right) \] (60)

By using the relation \( \mathbf{B} = \nabla \times \mathbf{A} \) and Stoke’s theorem, we find the vector potential to be
\[ \mathbf{A} = \frac{B \rho_o^2}{2\rho} \hat{\theta} = \frac{\Phi}{2\pi \rho} \hat{\theta} = A \hat{\theta} \] (61)
where $\Phi = \pi \rho_n^2 B$ is the total flux. Therefore, the Hamiltonian can be written as

$$H = \frac{1}{2m_e} \left( -i\hbar \nabla - \frac{e}{c} A \dot{\theta} \right) \cdot \left( -i\hbar \nabla - \frac{e}{c} A \dot{\theta} \right)$$

$$= \frac{-\hbar^2}{2m_e} \left( \nabla - \frac{ie}{\hbar c} A \dot{\theta} \right) \cdot \left( \nabla - \frac{ie}{\hbar c} A \dot{\theta} \right)$$

$$= \frac{-\hbar^2}{2m_e} \left( \rho \frac{\partial}{\partial \rho} + 2 \frac{\partial}{\partial z} + \dot{\theta} \frac{1}{\rho} \left[ \frac{\partial}{\partial \theta} - \frac{ie}{\hbar} \frac{\Phi}{2\pi} \right] \right) \cdot \left( \rho \frac{\partial}{\partial \rho} + \dot{z} \frac{\partial}{\partial z} + \dot{\theta} \frac{1}{\rho} \left[ \frac{\partial}{\partial \theta} - \frac{ie}{\hbar} \frac{\Phi}{2\pi} \right] \right)$$

$$H = \frac{-\hbar^2}{2m_e} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \left[ \frac{\partial}{\partial \theta} - \frac{ie}{\hbar} \frac{\Phi}{2\pi} \right]^2 + \frac{\partial^2}{\partial z^2} \right)$$

(62)

(63)

(64)

(65)

Defining $D_\Phi \equiv \frac{e}{\hbar c} \frac{\Phi}{2\pi}$, the Schrödinger equation reads

$$\frac{1}{\Theta} \left( \frac{\partial^2}{\partial \rho^2} - 2i D_\Phi \frac{\partial}{\partial \rho} \frac{\partial}{\partial \Theta} - D^2_\Phi \frac{\partial}{\partial \Theta} \right) \Theta = -m^2$$

(66)

Comparing this to equation 38, we see that both $\rho$ and $z$ terms are the same, while the $\theta$ term is altered by the factor $-iD_\Phi$. Therefore, in implementing separation of variables just as was done in part (a), the solutions $R(\rho)$ and $Z(z)$ will be exactly the same as those calculated above. The only difference will be that $\Theta(\theta)$ must now satisfy the following equation

$$\frac{1}{\Theta} \left( \frac{\partial^2}{\partial \rho^2} - 2i D_\Phi \frac{\partial}{\partial \rho} \frac{\partial}{\partial \Theta} - D^2_\Phi \frac{\partial}{\partial \Theta} \right) \Theta = -m^2$$

(67)

Attempting a solution of the form $\Theta = e^{i \theta}$, we find

$$\ell^2 - 2i D_\Phi \ell + \left( m^2 - D^2_\Phi \right) = 0$$

(68)

which yields

$$\ell = \frac{1}{2} \left( 2i D_\Phi \pm \left[ -4D^2_\Phi - 4(m^2 - D^2_\Phi) \right]^{1/2} \right)$$

(69)

$$\ell = i D_\Phi \pm \frac{1}{2} \sqrt{-4m^2}$$

(70)

$$\ell = i(D_\Phi \pm m)$$

(71)

Therefore, the solutions have the form

$$\Theta(\theta) = e^{i(D_\Phi \pm m)\theta}$$

(72)

However, imposing the periodicity constraint, we find that

$$e^{i(D_\Phi \pm m)\theta} = e^{i(D_\Phi \pm m)(\theta + 2\pi)}$$

(73)

$$\therefore m' = D_\Phi \pm m, \quad m' \in \mathbb{Z}$$

(74)

This leads to the result that $m = m' \mp D_\Phi$ is no longer necessarily an integer. Therefore, the energy eigenfunctions and eigenvalues are changed to
\[
\psi_{\ell mn}(\rho, \theta, z) = C \left( A_m J_m(k_{mn}\rho) + B_m N_m(k_{mn}\rho) \right) e^{im\theta} \sin \left( \frac{\ell \pi z}{L} \right)
\]

\[
E_{\ell mn} = \frac{\hbar^2}{2m_e} \left[ k_{mn}^2 + \left( \frac{\ell \pi}{L} \right)^2 \right]
\]

\[m = m' \mp D_\Phi\]

despite the electron not directly interacting with the magnetic field.

c) In order for the ground state to be unchanged when the magnetic field is applied, we require that

\[m = m' \mp D_\Phi = 0\]  \hspace{1cm} (75)

This implies a flux quantization

\[D_\Phi = \pm m'\]  \hspace{1cm} (76)

\[\frac{e}{\hbar c} \frac{\Phi}{2\pi} = m'\]  \hspace{1cm} (77)

\[\Phi = \frac{2\pi m'\hbar c}{e}, \quad m' = \pm 1, \pm 2, \ldots\]  \hspace{1cm} (78)